

CANCELLATION THEOREM FOR FRAMED MOTIVES OF ALGEBRAIC VARIETIES

A. ANANYEVSKIY, G. GARKUSHA, AND I. PANIN

ABSTRACT. The machinery of framed (pre)sheaves was developed by Voevodsky [V1]. Based on the theory, framed motives of algebraic varieties are introduced and studied in [GP1]. An analog of Voevodsky's Cancellation Theorem [V2] is proved in this paper for framed motives stating that a natural map of framed S^1 -spectra

$$M_{fr}(X)(n) \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)), \quad n \geq 0,$$

is a Nisnevich local stable equivalence, where $M_{fr}(X)(n)$ is the n th twisted framed motive of X . This result is reduced to the Cancellation Theorem for linear framed motives stating that the natural map of complexes of abelian groups

$$\mathbb{Z}F(\Delta^\bullet \times X, Y) \rightarrow \mathbb{Z}F((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)), \quad X, Y \in Sm/k,$$

is a quasi-isomorphism, where $\mathbb{Z}F(X, Y)$ is the group of stable linear framed correspondences in the sense of [GP1, GP3].

1. INTRODUCTION

In [V1] Voevodsky developed the machinery of framed correspondences and framed (pre)sheaves. Basing on this machinery, the theory of framed motives of algebraic varieties was introduced and studied in [GP1]. The framed motive of $X \in Sm/k$ is an explicitly constructed framed S^1 -spectrum $M_{fr}(X)$, which is connected and an Ω -spectrum in positive degrees (see [GP1] for details). Moreover, the shifts of the $M_{fr}(X)$ -s, $X \in Sm/k$, are compact generators of the associated compactly generated triangulated category of framed S^1 -spectra $SH_{S^1}^{fr}(k)$. The category $SH_{S^1}^{fr}(k)$ is the homotopy category of the category of framed S^1 -spectra $Sp_{S^1}^{fr}(k)$ with respect to the stable motivic model structure (see [GP1] for details).

The main object of [GP1] is the bispectrum

$$M_{fr}^{\mathbb{G}}(X) = (M_{fr}(X), M_{fr}(X)(1), M_{fr}(X)(2), \dots),$$

each term of which is a twisted framed motive of X and explicitly constructed structure maps

$$M_{fr}(X)(n) \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)), \quad n \geq 0.$$

Here $\mathbb{G} = \mathrm{Cyl}(t)/(-, pt)_+$ with $\mathrm{Cyl}(t)$ the mapping cylinder for the map $t : (-, pt)_+ \rightarrow (-, \mathbb{G}_m)_+$ sending pt to $1 \in \mathbb{G}_m$. The shifts of the $M_{fr}^{\mathbb{G}}(X)$ -s, $X \in Sm/k$, are compact generators of the associated compactly generated triangulated category of framed (S^1, \mathbb{G}) -bispectra $SH^{fr}(k)$. The category $SH^{fr}(k)$ is the homotopy category of the category of framed (S^1, \mathbb{G}) -bispectra $Sp_{S^1, \mathbb{G}}^{fr}(k)$ with respect to the stable motivic model structure (see [GP1] for details).

The main purpose of the paper is to prove the following (cf. Voevodsky [V2])

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Theorem A (Cancellation). *Let k be an infinite perfect field, $X \in Sm/k$ and $n \geq 0$. Then the following statements are true:*

(1) *the natural map of framed S^1 -spectra*

$$M_{fr}(X)(n) \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1))$$

is a Nisnevich local stable equivalence;

(2) *the induced map of framed S^1 -spectra*

$$M_{fr}(X)(n)_f \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)_f)$$

is a schemewise stable equivalence with $M_{fr}(X)(n)_f$ and $M_{fr}(X)(n+1)_f$ being framed Nisnevich local fibrant replacements $M_{fr}(X)(n)$ and $M_{fr}(X)(n+1)$ respectively.

As an application of Theorem A we prove the following

Theorem B. *Let k be an infinite perfect field, $X \in Sm/k$ and $n \geq 0$. Then the bispectrum*

$$M_{fr}^{\mathbb{G}}(X)_f = (M_{fr}(X)_f, M_{fr}(X)(1)_f, M_{fr}(X)(2)_f, \dots)$$

obtained from $M_{fr}^{\mathbb{G}}(X)$ by taking levelwise framed Nisnevich local fibrant replacements with structure maps those of Theorem A(2) is a motivically fibrant (S^1, \mathbb{G}) -bispectrum.

The motivic model category of framed S^1 -spectra $Sp_{S^1}^{fr}(k)$ has a natural Quillen pair of adjoint functors

$$- \boxtimes \mathbb{G} : Sp_{S^1}^{fr}(k) \rightleftarrows Sp_{S^1}^{fr}(k) : \underline{\mathrm{Hom}}(\mathbb{G}, -)$$

(see [GP1] for details). This Quillen pair induces adjoint functors on the homotopy category

$$- \boxtimes^L \mathbb{G} : SH_{S^1}^{fr}(k) \rightleftarrows SH_{S^1}^{fr}(k) : R\underline{\mathrm{Hom}}(\mathbb{G}, -).$$

The functor $- \boxtimes^L \mathbb{G}$ is also referred to as the *twist functor*. We also prove that the twist functor on $SH_{S^1}^{fr}(k)$ is fully faithful. More precisely, the following theorem is true.

Theorem C. *Let k be an infinite perfect field. Then the functor*

$$- \boxtimes^L \mathbb{G} : SH_{S^1}^{fr}(k) \rightarrow SH_{S^1}^{fr}(k)$$

is full and faithful.

The main strategy of proving Theorem A is to reduce it to the “Linear Cancellation Theorem”. In order to formulate it, recall from [GP1, GP3] that the category $\mathbb{Z}F_*(k)$ is an additive category whose objects are those of Sm/k and Hom-groups are defined as follows. We set for every $n \geq 0$ and $X, Y \in Sm/k$,

$$\mathbb{Z}F_n(X, Y) := \mathbb{Z}\mathrm{Fr}_n(X, Y) / \langle Z_1 \sqcup Z_2 - Z_1 - Z_2 \rangle,$$

where Z_1, Z_2 are supports of (level n) framed correspondences in the sense of Voevodsky [V1]. In other words, $\mathbb{Z}F_n(X, Y)$ is a free abelian group generated by the framed correspondences of level n with connected supports. We then set

$$\mathrm{Hom}_{\mathbb{Z}F_*(k)}(X, Y) := \bigoplus_{n \geq 0} \mathbb{Z}F_n(X, Y).$$

Given smooth varieties $X, Y \in Sm/k$ and $n \geq 0$, there is a canonical suspension morphism $\Sigma : \mathbb{Z}F_n(X, Y) \rightarrow \mathbb{Z}F_{n+1}(X, Y)$. We can stabilize in the Σ -direction to get an abelian group (see Definition 2.5)

$$\mathbb{Z}F(X, Y) := \mathrm{colim}(\mathbb{Z}F_0(k)(X, Y) \xrightarrow{\Sigma} \mathbb{Z}F_1(k)(X, Y) \xrightarrow{\Sigma} \dots).$$

There is a canonical morphism (see Definition 2.8 for more details), functorial in both arguments,

$$-\boxtimes(\mathrm{id}_{\mathbb{G}_m} - e_1) : \mathbb{Z}\mathbb{F}(X, Y) \rightarrow \mathbb{Z}\mathbb{F}(X \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$$

with $(\mathbb{G}_m, 1)$ the scheme $\mathbb{A}^1 - \{0\}$ pointed at 1.

The Linear Cancellation Theorem is formulated as follows.

Theorem D (Linear Cancellation). *Let k be an infinite perfect field and let X and Y be k -smooth schemes. Then*

$$-\boxtimes(\mathrm{id}_{\mathbb{G}_m} - e_1) : \mathbb{Z}\mathbb{F}(\Delta^\bullet \times X, Y) \rightarrow \mathbb{Z}\mathbb{F}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$$

is a quasi-isomorphism of complexes of abelian groups.

One of the main computational results of [GP3] says that homology of the complex $\mathbb{Z}\mathbb{F}(\Delta^\bullet \times -, Y)$ locally computes homology of the framed motive $M_{fr}(Y)$ of $Y \in Sm/k$. Moreover, the complex represents the “linear framed motive” of Y (see [GP3] for details).

Throughout the paper the base field k is infinite and perfect and Sm/k is the category of smooth separated schemes of finite type over the field k .

2. PRELIMINARIES

In this section we collect basic facts for framed correspondences. We start with preparations.

Let V be a scheme and Z be a closed subscheme. Recall that an *étale neighborhood* of Z in V is a triple $(W', \pi' : W' \rightarrow V, s' : Z \rightarrow W')$ satisfying the following conditions:

- (i) π' is an étale morphism;
- (ii) $\pi' \circ s'$ coincides with the inclusion $Z \hookrightarrow V$ (thus s' is a closed embedding).

A morphism between two étale neighborhoods $(W', \pi', s') \rightarrow (W'', \pi'', s'')$ of Z in V is a morphism $\rho : W' \rightarrow W''$ such that $\pi'' \circ \rho = \pi'$ and $\rho \circ s' = s''$. Note that such ρ is automatically étale by [EGA4, VI.4.7].

Definition 2.1 (Voevodsky [V1]). (I) Let Z be a closed subset in X . A *framing* of Z of level n is a collection $\varphi_1, \dots, \varphi_n$ of regular functions on X such that $\cap_{i=1}^n \{\varphi_i = 0\} = Z$. For a scheme X over S a framing of Z over S is a framing of Z such that the closed subsets $\{\varphi_i = 0\}$ do not contain the generic points of the fibers of $X \rightarrow S$.

(II) For k -smooth schemes X, Y over S and $n \geq 0$ an *explicit framed correspondence* Φ of level n consists of the following data:

- (1) a closed subset Z in \mathbb{A}_X^n which is finite over X ;
- (2) an étale neighborhood $p : U \rightarrow \mathbb{A}_X^n$ of Z ;
- (3) a framing $\varphi_1, \dots, \varphi_n$ of level n of Z in U over X ;
- (4) a morphism $g : U \rightarrow Y$.

The subset Z will be referred to as the *support* of the correspondence. We shall also write triples $\Phi = (U, \varphi, g)$ or quadruples $\Phi = (Z, U, \varphi, g)$ to denote explicit framed correspondences.

(III) Two explicit framed correspondences Φ and Φ' of level n are said to be *equivalent* if they have the same support and there exists an étale neighborhood V of Z in $U \times_{\mathbb{A}_X^n} U'$ such that on V , the morphism $g \circ pr$ agrees with $g' \circ pr'$ and $\varphi \circ pr$ agrees with $\varphi' \circ pr'$. A *framed correspondence of level n* is an equivalence class of explicit framed correspondences of level n .

We let $\mathrm{Fr}_n(X, Y)$ denote the set of framed correspondences from X to Y . We consider it as a pointed set with the distinguished point being the class 0_n of the explicit correspondence with $U = \emptyset$.

As an example, the set $\text{Fr}_0(X, Y)$ coincides with the set of pointed morphisms $X_+ \rightarrow Y_+$. In particular, for a connected scheme X one has

$$\text{Fr}_0(X, Y) = \text{Hom}_{\text{Sch}/S}(X, Y) \sqcup \{0_0\}.$$

If $f : X' \rightarrow X$ is a morphism of schemes and $\Phi = (U, \varphi, g)$ an explicit correspondence from X to Y then

$$f^*(\Phi) := (U' = U \times_X X', \varphi \circ pr, g \circ pr)$$

is an explicit correspondence from X' to Y .

From now on we shall only work with framed correspondences over smooth k -schemes Sm/k .

Remark 2.2. Let $\Phi = (Z, \mathbb{A}_X^n \xleftarrow{p} U, \varphi : U \rightarrow \mathbb{A}_k^n, g : U \rightarrow Y) \in \text{Fr}_n(X, Y)$ be an *explicit framed correspondence of level n* . It can more precisely be written in the form

$$((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) \in \text{Fr}_n(X, Y)$$

where

- ◇ $Z \subset \mathbb{A}_X^n$ is a closed subset finite over X ,
- ◇ an étale neighborhood $(\alpha_1, \alpha_2, \dots, \alpha_n), f) = p : U \rightarrow \mathbb{A}_k^n \times X$ of Z ,
- ◇ a framing $(\varphi_1, \varphi_2, \dots, \varphi_n) = \varphi : U \rightarrow \mathbb{A}_k^n$ of level n of Z in U over X ;
- ◇ a morphism $g : U \rightarrow Y$.

We shall usually drop $((\alpha_1, \alpha_2, \dots, \alpha_n), f)$ from notation and just write

$$(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) = ((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g).$$

The following definition is to describe compositions of framed correspondences.

Definition 2.3. Let X, Y and S be k -smooth schemes and let

$$a = ((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g)$$

be an explicit correspondence of level n from X to Y and let

$$b = ((\beta_1, \beta_2, \dots, \beta_m), f', Z', U', (\psi_1, \psi_2, \dots, \psi_m), g') \in \text{Fr}_m(Y, S)$$

be an explicit correspondence of level m from Y to S . We define their composition as an explicit correspondence of level $n + m$ from X to S by

$$((\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m), f, Z \times_Y Z', U \times_Y U', (\varphi_1, \varphi_2, \dots, \varphi_n, \psi_1, \psi_2, \dots, \psi_m), g').$$

Clearly, the composition of explicit correspondences respects the equivalence relation on them and defines associative maps

$$\text{Fr}_n(X, Y) \times \text{Fr}_m(Y, S) \rightarrow \text{Fr}_{n+m}(X, S).$$

Given $X, Y \in Sm/k$, denote by $\text{Fr}_*(X, Y)$ the set $\bigsqcup_n \text{Fr}_n(X, Y)$. The composition of framed correspondences defined above gives a category $\text{Fr}_*(k)$. Its objects are those of Sm/k and the morphisms are given by the sets $\text{Fr}_*(X, Y)$, $X, Y \in Sm/k$. Since the naive morphisms of schemes can be identified with certain framed correspondences of level zero, we get a canonical functor

$$Sm/k \rightarrow \text{Fr}_*(k).$$

One can easily see that for a framed correspondence $\Phi : X \rightarrow Y$ and a morphism $f : X' \rightarrow X$, one has $f^*(\Phi) = \Phi \circ f$.

Definition 2.4. Let X, Y, S and T be smooth schemes. There is an *external product*

$$\mathrm{Fr}_n(X, Y) \times \mathrm{Fr}_m(S, T) \xrightarrow{-\boxtimes-} \mathrm{Fr}_{n+m}(X \times S, Y \times T)$$

given by

$$\begin{aligned} &((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) \boxtimes ((\beta_1, \beta_2, \dots, \beta_m), f', Z', U', (\psi_1, \psi_2, \dots, \psi_m), g') = \\ &((\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m), f \times f', Z \times Z', U \times U', (\varphi_1, \varphi_2, \dots, \varphi_n, \psi_1, \psi_2, \dots, \psi_m), g \times g'). \end{aligned}$$

For the constant morphism $c: \mathbb{A}^1 \rightarrow \mathrm{pt}$, we set (following Voevodsky [V1])

$$\Sigma = -\boxtimes(t, c, \{0\}, \mathbb{A}^1, t, c): \mathrm{Fr}_n(X, Y) \rightarrow \mathrm{Fr}_{n+1}(X, Y)$$

and refer to it as the *suspension*.

Also, following Voevodsky [V1], one puts

$$\mathrm{Fr}(X, Y) = \mathrm{colim}(\mathrm{Fr}_0(X, Y) \xrightarrow{\Sigma} \mathrm{Fr}_1(X, Y) \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} \mathrm{Fr}_n(X, Y) \xrightarrow{\Sigma} \dots)$$

and refer to it as the *set stable framed correspondences*. The above external product induces external products

$$\begin{aligned} \mathrm{Fr}_n(X, Y) \times \mathrm{Fr}(S, T) &\xrightarrow{-\boxtimes-} \mathrm{Fr}(X \times S, Y \times T), \\ \mathrm{Fr}(X, Y) \times \mathrm{Fr}_0(S, T) &\xrightarrow{-\boxtimes-} \mathrm{Fr}(X \times S, Y \times T). \end{aligned}$$

Recall now the definition of the *category of linear framed correspondences* $\mathbb{Z}\mathrm{F}_*(k)$.

Definition 2.5. (see [GP1, p. 23]) Let X and Y be smooth schemes. Denote by

- ◇ $\mathbb{Z}\mathrm{Fr}_n(X, Y) := \widetilde{\mathbb{Z}}[\mathrm{Fr}_n(X, Y)] = \mathbb{Z}[\mathrm{Fr}_n(X, Y)]/\mathbb{Z} \cdot 0_n$, i.e the free abelian group generated by the set $\mathrm{Fr}_n(X, Y)$ modulo $\mathbb{Z} \cdot 0_n$;
- ◇ $\mathbb{Z}\mathrm{F}_n(X, Y) := \mathbb{Z}\mathrm{Fr}_n(X, Y)/A$, where A is a subgroup generated by the elements

$$\begin{aligned} &(Z \sqcup Z', U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) - \\ &-(Z, U \setminus Z', (\varphi_1, \varphi_2, \dots, \varphi_n)|_{U \setminus Z'}, g|_{U \setminus Z'}) - (Z', U \setminus Z, (\varphi_1, \varphi_2, \dots, \varphi_n)|_{U \setminus Z}, g|_{U \setminus Z}). \end{aligned}$$

We shall also refer to the latter relation as the *additivity property for supports*. In other words, it says that a framed correspondence in $\mathbb{Z}\mathrm{F}_n(X, Y)$ whose support is a disjoint union $Z \sqcup Z'$ equals the sum of the framed correspondences with supports Z and Z' respectively. Note that $\mathbb{Z}\mathrm{F}_n(X, Y)$ is $\mathbb{Z}[\mathrm{Fr}_n(X, Y)]$ modulo the subgroup generated by the elements as above, because $0_n = 0_n + 0_n$ in this quotient group, hence 0_n equals zero. Indeed, it is enough to observe that the support of 0_n equals $\emptyset \sqcup \emptyset$ and then apply the above relation to this support.

The elements of $\mathbb{Z}\mathrm{F}_n(X, Y)$ are called *linear framed correspondences of level n* or just *linear framed correspondences*.

Denote by $\mathbb{Z}\mathrm{F}_*(k)$ an additive category whose objects are those of Sm/k with Hom-groups defined as

$$\mathrm{Hom}_{\mathbb{Z}\mathrm{F}_*(k)}(X, Y) = \bigoplus_{n \geq 0} \mathbb{Z}\mathrm{F}_n(X, Y).$$

The composition is induced by the composition in the category $\mathrm{Fr}_*(k)$.

There is a functor $\mathrm{Sm}/k \rightarrow \mathbb{Z}\mathrm{F}_*(k)$ which is the identity on objects and which takes a regular morphism $f: X \rightarrow Y$ to the linear framed correspondence $1 \cdot (X, X \times \mathbb{A}^0, pr_{\mathbb{A}^0}, f \circ pr_X) \in \mathbb{Z}\mathrm{F}_0(k)$.

Definition 2.6. Let X, Y, S and T be schemes. The external product from Definition 2.4 induces a unique external product

$$\mathbb{Z}F_n(X, Y) \times \mathbb{Z}F_m(S, T) \xrightarrow{-\boxtimes-} \mathbb{Z}F_{n+m}(X \times S, Y \times T)$$

such that for any elements $a \in \text{Fr}_n(X, Y)$ and $b \in \text{Fr}_m(S, T)$ one has $1 \cdot a \boxtimes 1 \cdot b = 1 \cdot (a \boxtimes b) \in \mathbb{Z}F_{n+m}(X \times S, Y \times T)$.

For the constant morphism $c: \mathbb{A}^1 \rightarrow \text{pt}$, we set

$$\Sigma := - \boxtimes 1 \cdot (t, c, \{0\}, \mathbb{A}^1, t, c): \mathbb{Z}F_n(X, Y) \rightarrow \mathbb{Z}F_{n+1}(X, Y)$$

and refer to it as the *suspension*.

Definition 2.7. For any k -smooth variety X there is a presheaf $\mathbb{Z}F_*(-, Y)$ on the category $\mathbb{Z}F_*(k)$ represented by Y . The main $\mathbb{Z}F_*(k)$ -presheaf of this paper we are interested in is defined as

$$\mathbb{Z}F(-, Y) = \text{colim}(\mathbb{Z}F_0(-, Y) \xrightarrow{\Sigma} \mathbb{Z}F_1(-, Y) \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} \mathbb{Z}F_n(-, Y) \xrightarrow{\Sigma} \dots).$$

For a k -smooth variety X , elements of $\mathbb{Z}F(X, Y)$ are also called *stable linear framed correspondences*. Stable linear framed correspondences *do not form* morphisms of a category.

The main $\mathbb{Z}F_*(k)$ -presheaf of simplicial abelian groups we are interested in is defined as $\mathbb{Z}F(\Delta^\bullet \times X \times -, Y)$.

Definition 2.8. Let X and Y be k -smooth schemes and let (S, s) and (S', s') be k -smooth pointed schemes.

- ◊ Denote by $e_s: S \rightarrow \text{pt} \xrightarrow{s} S$ the idempotent given by the composition of the constant map and the embedding of s into S .
- ◊ Define $\mathbb{Z}F(X \wedge (S, s), Y \wedge (S, s))$ as a subgroup of $\mathbb{Z}F(X \times S, Y \times S)$ consisting of all a such that $a \circ (\text{id}_X \times e_s) = (\text{id}_Y \times e_s) \circ a = 0$. Note that these equalities are equivalent to

$$a \circ (\text{id}_X \boxtimes (\text{id}_S - e_s)) = (\text{id}_Y \boxtimes (\text{id}_S - e_s)) \circ a = a.$$

- ◊ Define $\mathbb{Z}F(X \wedge (S, s) \wedge (S', s'), Y \wedge (S, s) \wedge (S', s'))$ as a subgroup of $\mathbb{Z}F(X \times S \times S', Y \times S \times S')$ consisting of all a such that

$$a \circ (\text{id}_X \boxtimes (\text{id}_S - e_s) \boxtimes (\text{id}_{S'} - e_{s'})) = (\text{id}_Y \boxtimes (\text{id}_S - e_s) \boxtimes (\text{id}_{S'} - e_{s'})) \circ a = a.$$

We should mention that the preceding definition is necessary for the formulation of Theorem D (“Linear Cancellation”).

Theorem D. Let X and Y be k -smooth schemes. Then

$$-\boxtimes (\text{id}_{\mathbb{G}_m} - e_1): \mathbb{Z}F(\Delta^\bullet \times X, Y) \rightarrow \mathbb{Z}F((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$$

is a quasi-isomorphism of complexes abelian groups.

The theorem will be proved in Section 7.

3. THEOREM A, THEOREM B AND THEOREM C

Before proving Theorem A we recall some definitions and constructions for framed motives. We adhere to [GP1].

Recall that the category $sPre_\bullet^{fr}(k)$ of pointed simplicial framed presheaves consists of contravariant functors \mathcal{F} from $\text{Fr}_*(k)$ to pointed simplicial sets \mathbb{S}_\bullet such that $\mathcal{F}(\emptyset) = \text{pt}$. The category of S^1 -spectra associated with $sPre_\bullet^{fr}(k)$ is denoted by $Sp_{S^1}^{fr}(k)$. It has a stable motivic

projective model category structure whose homotopy category is denoted by $SH_{S^1}^{fr}(k)$. There is a canonical Quillen pair

$$\Phi : Sp_{S^1}(k) \rightleftarrows Sp_{S^1}^{fr}(k) : \Psi,$$

where $Sp_{S^1}(k)$ is the category of presheaves of S^1 -spectra equipped with the stable projective motivic model structure. The Quillen pair induces an adjoint pair of triangulated functors

$$\Phi : SH_{S^1}(k) \rightleftarrows SH_{S^1}^{fr}(k) : \Psi$$

between triangulated categories.

Given a finite pointed set $(K, *)$ and a scheme X , we denote by $X \otimes K$ the unpointed scheme $X \sqcup \dots \sqcup X$, where the coproduct is indexed by the non-based elements in K . By the Additivity Theorem of [GP1] the Γ -space in the sense of Segal [Seg]

$$K \in \Gamma^{op} \mapsto C_* Fr(U, X \otimes K) := Fr(U \times \Delta^\bullet, X \otimes K)$$

is special.

Definition 3.1 (see [GP1]). The *framed motive* $M_{fr}(X)$ of a smooth algebraic variety $X \in Sm/k$ is the Segal S^1 -spectrum $(C_* Fr(-, X), C_* Fr(-, X \otimes S^1), C_* Fr(-, X \otimes S^2), \dots)$ associated with the special Γ -space $K \in \Gamma^{op} \mapsto C_* Fr(-, X \otimes K)$.

The framed motive $M_{fr}(X) \in Sp_{S^1}^{fr}(k)$ is functorial in framed correspondences of level zero. Moreover, $\{M_{fr}(X)\}_{X \in Sm/k}$ are compact generators of $SH_{S^1}^{fr}(k)$. By the Resolution Theorem of [GP1] every motivic fibrant replacement $M_{fr}(X) \rightarrow M_{fr}(X)_f$ of $M_{fr}(X)$ in $Sp_{S^1}^{fr}(k)$ is a stable Nisnevich local equivalence in $Sp_{S^1}(k)$ (over perfect fields).

Denote by \mathbb{G} the pointed simplicial presheaf which is termwise

$$(-, \mathbb{G}_m)_+, (-, \mathbb{G}_m)_+ \vee (-, pt)_+, (-, \mathbb{G}_m)_+ \vee (-, pt)_+ \vee (-, pt)_+, \dots$$

As a pointed motivic space \mathbb{G} is $Cyl(\mathbf{t})/(-, pt)_+$ with $Cyl(\mathbf{t})$ the mapping cylinder for the map $\mathbf{t} : (-, pt)_+ \rightarrow (-, \mathbb{G}_m)_+$ sending pt to $1 \in \mathbb{G}_m$. By $\mathbb{G}_m^{\wedge 1}$ we mean the simplicial object in $Fr_0(k)$ which is termwise

$$\mathbb{G}_m, \mathbb{G}_m \sqcup pt, \mathbb{G}_m \sqcup pt \sqcup pt, \dots$$

Applying $M_{fr}(X \times -)$ to $\mathbb{G}_m^{\wedge 1}$ and realizing by taking diagonals, one gets a framed S^1 -spectrum $M_{fr}(X \times \mathbb{G}_m^{\wedge 1})$. We shall also denote it by $M_{fr}(X)(1)$. The n th iteration gives the spectrum $M_{fr}(X \times \mathbb{G}_m^{\wedge n})$, also denoted by $M_{fr}(X)(n)$. The nearest aim is to define the (S^1, \mathbb{G}) -bispectrum $M_{fr}^{\mathbb{G}}(X)$. Another way of defining the (S^1, \mathbb{G}) -bispectrum $M_{fr}^{\mathbb{G}}(X)$ is given in Appendix B.

We construct a map in $Sp_{S^1}^{fr}(k)$

$$a_0 : M_{fr}(X) \rightarrow \underline{\text{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1}))$$

as follows. It is uniquely determined by a map

$$\beta : M_{fr}(X) \rightarrow M_{fr}(X \times \mathbb{G}_m^{\wedge 1})(- \times \mathbb{G}_m)$$

and a homotopy

$$h : M_{fr}(X) \rightarrow M_{fr}(X \times \mathbb{G}_m^{\wedge 1})(- \times pt)^I \quad (1)$$

such that $d_0 h = f^* \beta$ and $d_1 h$ factors through the distinguished point levelwise. Here $f : pt \rightarrow \mathbb{G}_m$ is a morphism of schemes such that $f(pt) = 1$.

We set the map β to be the composition

$$M_{fr}(X) \xrightarrow{- \boxtimes \mathbb{G}_m} M_{fr}(X \times \mathbb{G}_m)(- \times \mathbb{G}_m) \xrightarrow{p} M_{fr}(X \times \mathbb{G}_m^{\wedge 1})(- \times \mathbb{G}_m),$$

where p is a natural map, induced by the simplicial map of simplicial objects $\mathbb{G}_m \rightarrow \mathbb{G}_m^{\wedge 1}$ in $\text{Fr}_0(k)$ (we consider \mathbb{G}_m as a simplicial scheme in a trivial way).

One has a commutative square for any $W \in \text{Sm}/k$

$$\begin{array}{ccc} C_*Fr(W \times \mathbb{G}_m, X \times \mathbb{G}_m) & \xrightarrow{C_*Fr(1_W \times f, 1_{X \times \mathbb{G}_m})} & C_*Fr(W \times pt, X \times \mathbb{G}_m) \\ \uparrow -\boxtimes \mathbb{G}_m & & \uparrow C_*Fr(1_W \times pt, 1_X \times f) \\ C_*Fr(W, X) & \xrightarrow{-\boxtimes pt} & C_*Fr(W \times pt, X \times pt). \end{array}$$

On the other hand, there is a commutative diagram

$$\begin{array}{ccc} C_*Fr(W \times pt, X \times \mathbb{G}_m) & \xrightarrow{p} & C_*Fr(W \times pt, X \times \mathbb{G}_m^{\wedge 1}) \\ \uparrow f_* & & \uparrow f_* \\ C_*Fr(W \times pt, X \times pt) & \xrightarrow{p'} & P(C_*Fr(W \times pt, (X \times pt) \otimes S^1)) \end{array}$$

Here the right lower corner stands for the simplicial path space of $C_*Fr(W \times pt, (X \times pt) \otimes S^1)$ (see [GP1, Section 7] for details). Recall that the *path space* PX of a simplicial object $X : \Delta^{\text{op}} \rightarrow \mathcal{D}$ in a category \mathcal{D} is defined as the composition of X with the shift functor $P : \Delta \rightarrow \Delta$ that takes $[n]$ to $[n+1]$ (by mapping i to $i+1$). By [Wal, 1.5.1] there is a canonical contraction of this space into the set of its zero simplices regarded as a constant simplicial set. Since $P(C_*Fr(W \times pt, (X \times pt) \otimes S^1))$ has only one zero simplex, it follows that there is a canonical simplicial homotopy

$$H : P(C_*Fr(W \times pt, (X \times pt) \otimes S^1)) \rightarrow P(C_*Fr(W \times pt, (X \times pt) \otimes S^1))^I$$

such that $d_0 H = 1$ and $d_1 H = \text{const.}$

Now the map $h(1)$ is induced by the composite map

$$\begin{array}{ccc} & & C_*Fr(W \times pt, X \times \mathbb{G}_m^{\wedge 1})^I \\ & & \uparrow f_*^I \\ P(C_*Fr(W \times pt, (X \times pt) \otimes S^1)) & \xrightarrow{H} & P(C_*Fr(W \times pt, (X \times pt) \otimes S^1))^I \\ \uparrow p' & & \\ C_*Fr(W, X) & \xrightarrow{-\boxtimes pt} & C_*Fr(W \times pt, X \times pt) \end{array}$$

(the same composite map is similarly defined on each space of the spectrum $M_{fr}(X)$). The desired map $a_0 : M_{fr}(X) \rightarrow \underline{\text{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1}))$ is constructed. Note that a_0 is functorial in framed correspondences of level zero. Each map of spectra

$$a_n : M_{fr}(X)(n) \rightarrow \underline{\text{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)), \quad n \geq 0, \quad (2)$$

is constructed similar to a_0 if we replace X with $X \times \mathbb{G}_m^{\wedge n}$ and realize by taking diagonals.

Definition 3.2. The (S^1, \mathbb{G}) -bispectrum $M_{fr}^{\mathbb{G}}(X)$ is defined as

$$(M_{fr}(X), M_{fr}(X \times \mathbb{G}_m^{\wedge 1}), M_{fr}(X \times \mathbb{G}_m^{\wedge 2}), \dots)$$

together with the structure morphisms a_n -s. Another way of defining the a_n -s is given in Appendix B.

We shall prove below (see the proof of Theorem A) that each a_n is a Nisnevich local stable equivalence of spectra, but first let us discuss further useful spectra.

Denote by $\mathbb{Z}\mathrm{Fr}_*^{S^1}(X)$ $X \in \mathrm{Sm}/k$ the Segal S^1 -spectrum $(\mathbb{Z}\mathrm{Fr}_*(-, X), \mathbb{Z}\mathrm{Fr}_*(-, X \otimes S^1), \dots)$. Denote by $EM(\mathbb{Z}\mathrm{F}_*(-, X))$ the Segal S^1 -spectrum $(\mathbb{Z}\mathrm{F}_*(-, X), \mathbb{Z}\mathrm{F}_*(-, X \otimes S^1), \dots)$. The equalities $\mathbb{Z}\mathrm{F}_*(-, X \sqcup X') = \mathbb{Z}\mathrm{F}_*(-, X) \oplus \mathbb{Z}\mathrm{F}_*(-, X')$ show that the Γ -space $(K, *) \mapsto \mathbb{Z}\mathrm{F}_*(U, X \otimes K)$ corresponds to the abelian group $\mathbb{Z}\mathrm{F}_*(U, X)$. Hence $EM(\mathbb{Z}\mathrm{F}_*(-, X))$ is the Eilenberg–Mac Lane spectrum for $\mathbb{Z}\mathrm{F}_*(-, X)$. The Γ -space morphism $[(K, *) \mapsto \mathbb{Z}\mathrm{Fr}_*(-, X \otimes K)] \rightarrow [(K, *) \mapsto \mathbb{Z}\mathrm{F}_*(-, X \otimes K)]$ induces a morphism of framed S^1 -spectra

$$\lambda_X : \mathbb{Z}\mathrm{Fr}_*^{S^1}(X) \rightarrow EM(\mathbb{Z}\mathrm{F}_*(-, X))$$

Also, denote by $\mathbb{Z}M_{fr}(X)$, $X \in \mathrm{Sm}/k$, the Segal S^1 -spectrum $(C_*\mathbb{Z}\mathrm{Fr}(-, X), C_*\mathbb{Z}\mathrm{Fr}(-, X \otimes S^1), \dots)$. Denote by $LM_{fr}(X)$ the Segal S^1 -spectrum $EM(\mathbb{Z}\mathrm{F}(\Delta^\bullet \times -, X)) = (\mathbb{Z}\mathrm{F}(\Delta^\bullet \times -, X), \mathbb{Z}\mathrm{F}(\Delta^\bullet \times -, X \otimes S^1), \dots)$. The above arguments show that $LM_{fr}(X)$ is the Eilenberg–Mac Lane spectrum for $\mathbb{Z}\mathrm{F}(\Delta^\bullet \times -, X)$ and one has a natural morphism

$$l_X : \mathbb{Z}M_{fr}(X) \rightarrow LM_{fr}(X)$$

of framed S^1 -spectra.

Note that homotopy groups of $LM_{fr}(X) = EM(\mathbb{Z}\mathrm{F}(\Delta^\bullet \times -, X))$ are equal to homology groups of the complex $\mathbb{Z}\mathrm{F}(\Delta^\bullet \times -, X)$. By [Sch, §II.6.2] homotopy groups $\pi_*(\mathbb{Z}M_{fr}(X)(U))$ of $\mathbb{Z}M_{fr}(X)$ evaluated at $U \in \mathrm{Sm}/k$ are homology groups $H_*(M_{fr}(X)(U))$ of $M_{fr}(X)(U)$.

As above we can define S^1 -spectra $LM_{fr}(X \times \mathbb{G}_m^{\wedge n})$ -s together with morphisms of framed spectra

$$c_n : LM_{fr}(X \times \mathbb{G}_m^{\wedge n}) \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge n+1})), \quad n \geq 0.$$

We also refer the reader to Appendix B for another way of defining the c_n -s.

Though $\mathbb{Z}M_{fr}(X)(U)$ is a linear S^1 -spectrum, its homotopy groups are hard to compute. But if U is smooth local Henselian, then $\pi_*(\mathbb{Z}M_{fr}(X)(U)) = H_*(\mathbb{Z}\mathrm{F}(\Delta^\bullet \times U, X))$. More precisely, the following result is true.

Theorem 3.3 (see [GP3]). *The natural morphisms of framed S^1 -spectra*

$$\lambda_X : \mathbb{Z}\mathrm{Fr}_*^{S^1}(X) \rightarrow EM(\mathbb{Z}\mathrm{F}_*(-, X)) \quad \text{and} \quad l_X : \mathbb{Z}M_{fr}(X) \rightarrow LM_{fr}(X)$$

are Nisnevich local stable equivalences. In particular, if U is smooth local Henselian, then $\pi_(\mathbb{Z}M_{fr}(X)(U)) = H_*(\mathbb{Z}\mathrm{F}(\Delta^\bullet \times U, X))$.*

Definition 3.4. (I) Following [GP1] $\mathcal{X} \in Sp_{S^1}^{fr}(k)$ is *BCD-local* if it is \mathbb{A}^1 -invariant, takes the level one framed correspondence $\sigma_X = (\{0\}, \mathbb{A}_X^1 \xrightarrow{\mathrm{id}} \mathbb{A}_X^1, pr_X) : X \rightarrow X$ with $X \in \mathrm{Sm}/k$ to a stable weak equivalence and the natural map of spectra

$$\mathcal{X}(U \sqcup V) \rightarrow \mathcal{X}(U) \times \mathcal{X}(V)$$

is a stable weak equivalence of spectra for all $U, V \in \mathrm{Sm}/k$.

(II) A framed presheaf of abelian groups \mathcal{F} is *quasi-stable* if $\mathcal{F}(\sigma_X)$ is an isomorphism for every $X \in \mathrm{Sm}/k$. \mathcal{F} is *radditive* if the natural homomorphism

$$\mathcal{F}(U \sqcup V) \rightarrow \mathcal{F}(U) \times \mathcal{F}(V)$$

is an isomorphism for all $U, V \in \mathrm{Sm}/k$ and $\mathcal{F}(\emptyset) = 0$.

Note that $\mathcal{X} \in Sp_{S^1}^{fr}(k)$ is *BCD-local* if and only if each homotopy presheaf $\pi_n(\mathcal{X})$, $n \in \mathbb{Z}$, is \mathbb{A}^1 -invariant, quasi-stable and radditive.

Below we shall need the following

Lemma 3.5. *Suppose $\rho : \mathcal{X} \rightarrow \mathcal{Y}$ is a stable Nisnevich local weak equivalence of framed BCD -local S^1 -spectra. Then $\rho_* : \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{X}) \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{Y})$ is a stable Nisnevich local weak equivalence.*

Proof. We have to show that

$$\rho_{U \boxtimes \mathbb{G}_m} : \mathcal{X}(U \boxtimes \mathbb{G}_m) \rightarrow \mathcal{Y}(U \boxtimes \mathbb{G}_m)$$

is a stable weak equivalence of spectra for every local smooth Henselian U . Consider the presheaf of S^1 -spectra

$$U \in \mathrm{Sm}/k \mapsto \mathrm{cone}(\rho_{U \boxtimes \mathbb{G}_m}).$$

The spectra $\mathcal{X}(- \boxtimes \mathbb{G}_m)$, $\mathcal{Y}(- \boxtimes \mathbb{G}_m)$ are BCD -local. Therefore their presheaves of homotopy groups are quasi-stable, radditive \mathbb{A}^1 -invariant framed presheaves. It follows that the presheaves of homotopy groups of $\mathrm{cone}(\rho_{- \boxtimes \mathbb{G}_m})$ are quasi-stable, radditive \mathbb{A}^1 -invariant framed presheaves. Thus the presheaves of homotopy groups of $\mathrm{cone}(\rho_{- \boxtimes \mathbb{G}_m})$, $\mathcal{X}(- \boxtimes \mathbb{G}_m)$, $\mathcal{Y}(- \boxtimes \mathbb{G}_m)$ are quasi-stable, \mathbb{A}^1 -invariant $\mathbb{Z}F_*(k)$ -presheaves (see [GP2, Introduction]).

By [GP2, 2.15(3')] for any quasi-stable, \mathbb{A}^1 -invariant $\mathbb{Z}F_*(k)$ -presheaf F and every smooth local Henselian U the pull-back map $F(U) \rightarrow F(\mathrm{Spec} k(U))$ is injective. By [GP2, 2.15(1)-(2)] the presheaf $F|_{\mathbb{A}_k^1}$ is a Zariski sheaf. Using [GP2, 2.15(5)] applied to $X = \mathbb{A}_k^1$, one shows that for any open W in \mathbb{A}_k^1 one has $F^{\mathrm{nis}}(W) = F(W)$. In particular, $F^{\mathrm{nis}}(\mathbb{G}_{m,k}) = F(\mathbb{G}_{m,k})$.

Hence for every smooth local Henselian U , the homotopy groups $\pi_*(\mathrm{cone}(\rho_{U \boxtimes \mathbb{G}_m}))$ are embedded into $\pi_*(\mathrm{cone}(\rho_{\mathrm{Spec} k(U) \boxtimes \mathbb{G}_m}))$ and for any k -smooth irreducible variety V one has

$$\begin{aligned} \pi_*(\mathcal{X})(\mathrm{Spec} k(V) \boxtimes \mathbb{G}_m) &= \pi_*^{\mathrm{nis}}(\mathcal{X})(\mathrm{Spec} k(V) \boxtimes \mathbb{G}_m) = \\ \pi_*^{\mathrm{nis}}(\mathcal{Y})(\mathrm{Spec} k(V) \boxtimes \mathbb{G}_m) &= \pi_*(\mathcal{Y})(\mathrm{Spec} k(V) \boxtimes \mathbb{G}_m). \end{aligned}$$

Thus for every smooth local Henselian U , $\pi_*(\mathrm{cone}(\rho_{\mathrm{Spec} k(U) \boxtimes \mathbb{G}_m})) = 0$, and hence $\pi_*(\mathrm{cone}(\rho_{U \boxtimes \mathbb{G}_m})) = 0$. Therefore $\rho_{U \boxtimes \mathbb{G}_m}$ is a stable equivalence whenever U is local smooth Henselian. \square

By the Resolution Theorem of [GP1] one can find for every framed BCD -local S^1 -spectrum \mathcal{G} a Nisnevich local fibrant replacement

$$\alpha : \mathcal{G} \rightarrow \mathcal{G}_f$$

such that α is a map in $Sp_{S^1}^{fr}(k)$ which induces an isomorphism $\pi_*^{\mathrm{nis}}(\alpha)$ on the Nisnevich stable homotopy sheaves. Moreover, α is functorial in \mathcal{G} . In particular, we can find $M_{fr}(X)(n)_f$ for every $n \geq 0$. Each map (2) induces a map of framed spectra

$$b_n : M_{fr}(X)(n)_f \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)_f), \quad n \geq 0,$$

such that the square

$$\begin{array}{ccc} M_{fr}(X)(n) & \xrightarrow{a_n} & \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)) \\ \alpha \downarrow & & \downarrow \underline{\mathrm{Hom}}(\mathbb{G}, \alpha) \\ M_{fr}(X)(n)_f & \xrightarrow{b_n} & \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)_f) \end{array} \quad (3)$$

is commutative.

We are now in a position to prove Theorem A.

Theorem A (Cancellation). *Let k be an infinite perfect field, $X \in \mathcal{S}m/k$ and $n \geq 0$. Then the following statements are true:*

(1) *the natural map of framed S^1 -spectra*

$$a_n : M_{fr}(X)(n) \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1))$$

is a Nisnevich local stable equivalence;

(2) *the induced map of framed S^1 -spectra*

$$b_n : M_{fr}(X)(n)_f \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)_f)$$

is a schemewise stable equivalence with $M_{fr}(X)(n)_f$ and $M_{fr}(X)(n+1)_f$ being framed Nisnevich local fibrant replacements $M_{fr}(X)(n)$ and $M_{fr}(X)(n+1)$ respectively.

Proof. Since $M_{fr}(X)(n), M_{fr}(X)(n)_f$ are BCD -local and we have a commutative diagram with homotopy fiber rows in $Sp_{S^1}(k)$

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n)) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m, M_{fr}(X)(n)) & \longrightarrow & M_{fr}(X)(n) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n)_f) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m, M_{fr}(X)(n)_f) & \longrightarrow & M_{fr}(X)(n)_f \end{array}$$

then the vertical maps of diagram (3) are Nisnevich local stable equivalences (we use here Lemma 3.5). It follows that a_n is a Nisnevich local stable equivalence if and only if so is b_n . The latter is equivalent to saying that b_n is a schemewise equivalence, because $M_{fr}(X)(n)_f$ and $\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)_f)$ are motivically fibrant by the Resolution Theorem of [GP1].

It is enough to prove that

$$b_0 : M_{fr}(X)_f \rightarrow \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(1)_f)$$

is a schemewise equivalence of spectra. Indeed, consider a commutative diagram of homotopy cofiber sequences in $Sp_{S^1}(k)$

$$\begin{array}{ccccc} M_{fr}(X)(n-1)_f & \longrightarrow & M_{fr}(X \times \mathbb{G}_m)(n-1)_f & \longrightarrow & M_{fr}(X)(n)_f \\ b_{n-1} \downarrow & & b_{n-1} \downarrow & & \downarrow b_n \\ \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n)_f) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m)(n)_f) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X)(n+1)_f) \end{array}$$

with $n \geq 1$. If b_{n-1} is a schemewise equivalence of spectra, then so is b_n by [Hir, 13.5.10].

Thus using induction in n , it suffices to verify that b_0 is a schemewise equivalence of spectra. As we have mentioned above this is equivalent to saying that a_0 is a Nisnevich local equivalence of spectra.

By the stable Whitehead theorem [Sch, II.6.30] a_0 is a stable local Nisnevich equivalence if and only if so is

$$a_0 : \mathbb{Z}M_{fr}(X) \rightarrow \mathbb{Z}[\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1}))].$$

Consider a commutative diagram of homotopy fiber sequences in $Sp_{S^1}(k)$

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m, M_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & M_{fr}(X \times \mathbb{G}_m^{\wedge 1}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}[\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1}))] & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathbb{Z}M_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & \mathbb{Z}M_{fr}(X \times \mathbb{G}_m^{\wedge 1}) \\
\downarrow \ell_X & & \downarrow & & \downarrow l_X \\
\underline{\mathrm{Hom}}(\mathbb{G}, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & \underline{\mathrm{Hom}}(\mathbb{G}_m, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})) & \longrightarrow & LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})
\end{array}$$

The arrow l_X is a stable local weak equivalence of BCD -local spectra by Theorem 3.3, and hence so is the middle lower arrow by Lemma 3.5. It follows that ℓ_X is a stable local weak equivalence. Consider a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}M_{fr}(X) & \xrightarrow{a_0} & \mathbb{Z}[\underline{\mathrm{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge 1}))] \\
\downarrow l_X & & \downarrow \ell_X \\
LM_{fr}(X) & \xrightarrow{c_0} & \underline{\mathrm{Hom}}(\mathbb{G}, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})).
\end{array}$$

We have mentioned above that l_X, ℓ_X are stable local weak equivalences. It follows that a_0 is a stable local equivalence if and only if so is c_0 . We shall prove that c_0 is a sectionwise stable equivalence. To this end we need notation from Appendix A and Lemma A.1. Consider the commutative diagram from Lemma A.1.

$$\begin{array}{ccc}
LM_{fr}(X) & \xrightarrow{can'_* \circ (-\boxtimes(\mathrm{id}_{\mathbb{G}_m} - e_{\mathbb{G}_m}))} & \underline{\mathrm{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})) \\
\uparrow id & & \uparrow \underline{\mathrm{Hom}}((\mathbb{G}_m, 1), can'_*|_{LM_{fr}(X \wedge (\mathbb{G}_m, 1))}) \\
LM_{fr}(X) & \xrightarrow{-\boxtimes(\mathrm{id}_{\mathbb{G}_m} - e_{\mathbb{G}_m})} & \underline{\mathrm{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \wedge (\mathbb{G}_m, 1)))
\end{array}$$

Since $LM_{fr}(X), \underline{\mathrm{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \wedge (\mathbb{G}_m, 1)))$ are schemewise linear Ω -spectra, then homotopy groups $\pi_*(LM_{fr}(X))$ (respectively $\pi_*(\underline{\mathrm{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \wedge (\mathbb{G}_m, 1))))$ equal homology groups $H_*(\mathbb{Z}F(\Delta^\bullet \times -, X))$ (respectively $\pi_*(\underline{\mathrm{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \wedge (\mathbb{G}_m, 1)))) = H_*(\mathbb{Z}F((\Delta^\bullet \times -) \wedge (\mathbb{G}_m, 1), X \wedge (\mathbb{G}_m, 1)))$. Hence the bottom arrow $-\boxtimes(\mathrm{id}_{\mathbb{G}_m} - e_{\mathbb{G}_m})$ is a sectionwise stable equivalence by Theorem D.

The arrow $\underline{\mathrm{Hom}}((\mathbb{G}_m, 1), can'_*|_{LM_{fr}(X \wedge (\mathbb{G}_m, 1))})$ is a sectionwise stable equivalence by Lemma A.1. Hence the arrow $can'_* \circ (-\boxtimes(\mathrm{id}_{\mathbb{G}_m} - e_{\mathbb{G}_m}))$ is a sectionwise stable equivalence.

By Lemma A.1 one has $can'_* \circ (-\boxtimes(\mathrm{id}_{\mathbb{G}_m} - e_{\mathbb{G}_m})) = [in^* \circ \overline{can^* \circ (\mathrm{id}_{\mathbb{G}}^* - e_{\mathbb{G}}^*)}] \circ c_0$ and the morphism

$$\underline{\mathrm{Hom}}(\mathbb{G}, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})) \xrightarrow{in^* \circ \overline{can^* \circ (\mathrm{id}_{\mathbb{G}}^* - e_{\mathbb{G}}^*)}} \underline{\mathrm{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \times \mathbb{G}_m^{\wedge 1}))$$

is a sectionwise stable equivalence. Thus c_0 is a sectionwise stable equivalence. This completes the proof of Theorem A. \square

Theorem B. *Let k be an infinite perfect field, $X \in Sm/k$. Then the bispectrum*

$$M_{fr}^{\mathbb{G}}(X)_f = (M_{fr}(X)_f, M_{fr}(X)(1)_f, M_{fr}(X)(2)_f, \dots)$$

obtained from $M_{fr}^{\mathbb{G}}(X)$ by taking levelwise framed Nisnevich local fibrant replacements with structure maps b_n -s is a motivically fibrant (S^1, \mathbb{G}) -bispectrum.

Proof. By the Resolution Theorem of [GP1] each framed S^1 -spectrum $M_{fr}(X)(n)_f$ is motivically fibrant. By Theorem A each structure map b_n is a schemewise equivalence. We conclude that the bispectrum $M_{fr}^{\mathbb{G}}(X)_f$ is a motivically fibrant (S^1, \mathbb{G}) -bispectrum. \square

The motivic model category of framed S^1 -spectra $Sp_{S^1}^{fr}(k)$ has a natural Quillen pair of adjoint functors

$$- \boxtimes \mathbb{G} : Sp_{S^1}^{fr}(k) \rightleftarrows Sp_{S^1}^{fr}(k) : \underline{\text{Hom}}(\mathbb{G}, -)$$

(see [GP1] for details). This Quillen pair induces adjoint functors on the homotopy category

$$- \boxtimes^L \mathbb{G} : SH_{S^1}^{fr}(k) \rightleftarrows SH_{S^1}^{fr}(k) : R\underline{\text{Hom}}(\mathbb{G}, -).$$

The functor $- \boxtimes^L \mathbb{G}$ is also referred to as the *twist functor*. By construction, $M_{fr}(X) \boxtimes^L \mathbb{G}$ is canonically isomorphic to $M_{fr}(X)(1)$ for all $X \in Sm/k$ (see [GP1] for details).

We finish the section by proving Theorem C.

Theorem C. *Let k be an infinite perfect field. Then the twist functor*

$$- \boxtimes^L \mathbb{G} : SH_{S^1}^{fr}(k) \rightarrow SH_{S^1}^{fr}(k)$$

is full and faithful.

Proof. The twist functor $- \boxtimes^L \mathbb{G}$ is triangulated and preserves arbitrary coproducts. By [GP1, 6.15] $SH_{S^1}^{fr}(k)$ is a compactly generated triangulated category with $\mathcal{C} := \{M_{fr}(X)[\ell] \mid X \in Sm/k, \ell \in \mathbb{Z}\}$ a family of compact generators.

Theorem A and [GP1, 6.15] imply that our theorem is true for compact generators from \mathcal{C} . By using the five-lemma one can easily show that our theorem is also true for all compact objects. Since the twist functor is triangulated and preserves arbitrary coproducts, our proof now follows from the fact that every object of $SH_{S^1}^{fr}(k)$ is a homotopy colimit of compact objects. \square

4. USEFUL LEMMAS

In this section we discuss several useful \mathbb{A}^1 -homotopies and collect a number of facts used in the following sections. We start with some definitions and notation.

Definition 4.1. Let $\mathcal{F} : Sm/k \rightarrow Sets$ be a presheaf of sets. Let $X \in Sm/k$ be a smooth variety and $a, b \in \mathcal{F}(X)$ be two sections. We write $a \sim b$ if a and b are in the same connected component of the simplicial set $\mathcal{F}(\Delta^\bullet \times X)$. If $h \in \mathcal{F}(\Delta^1 \times X)$ is such that $\partial_0(h) = a$ and $\partial_1(h) = b$, then we will write $a \stackrel{h}{\sim} b$. In this case $a \sim b$.

Let $\mathcal{A} : Sm/k \rightarrow Ab$ be a presheaf of abelian groups. Let $X \in Sm/k$ be a smooth variety and $a, b \in \mathcal{A}(X)$ be two sections. We will write $a \sim b$ if the classes of a and b in $H_0(\mathcal{A}(\Delta^\bullet \times X))$ coincide. This is equivalent to saying that there is $h \in \mathcal{A}(\Delta^1 \times X)$ such that $\partial_0(h) = a$ and $\partial_1(h) = b$. For such an h we will write $a \stackrel{h}{\sim} b$.

Definition 4.2. Let \mathcal{F} and \mathcal{G} be two presheaves of sets on the category of k -smooth schemes and let $\varphi_0, \varphi_1 : \mathcal{F} \rightrightarrows \mathcal{G}$ be two morphisms. An \mathbb{A}^1 -homotopy between φ_0 and φ_1 is a morphism $H : \mathcal{F} \rightarrow \underline{\text{Hom}}(\mathbb{A}^1, \mathcal{G})$ such that $H_0 = \varphi_0$ and $H_1 = \varphi_1$. We will write $\varphi_0 \sim \varphi_1$ if there is an \mathbb{A}^1 -homotopy between φ_0 and φ_1 .

Let \mathcal{A} and \mathcal{B} be two presheaves of abelian groups on the category of k -smooth schemes and let $\varphi_0, \varphi_1 : \mathcal{A} \rightrightarrows \mathcal{B}$ be two morphisms. An \mathbb{A}^1 -homotopy between φ_0 and φ_1 is a morphism $H : \mathcal{A} \rightarrow \underline{\text{Hom}}(\mathbb{A}^1, \mathcal{B})$ of presheaves of abelian groups such that $H_0 = \varphi_0$ and $H_1 = \varphi_1$. If H is an \mathbb{A}^1 -homotopy between φ_0 and φ_1 , then we will write $\varphi_0 \stackrel{H}{\sim} \varphi_1$. If we do not specify an \mathbb{A}^1 -homotopy between φ_0 and φ_1 , then we will write $\varphi_0 \sim \varphi_1$.

If $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of presheaves of abelian groups, then there is a constant \mathbb{A}^1 -homotopy H_φ between φ and φ defined as follows. Given $a \in \mathcal{A}(X)$ set $H_\varphi(a) = pr_X^*(\varphi(a)) \in \mathcal{B}(X \times \mathbb{A}^1)$.

Lemma 4.3. *Let \mathcal{A} and \mathcal{B} be two presheaves of abelian groups on the category of k -smooth schemes and let $\varphi_0, \varphi_1 : \mathcal{A} \rightrightarrows \mathcal{B}$ be two morphisms such that $\varphi_0 \sim \varphi_1$. Then the induced morphisms*

$$\varphi_0, \varphi_1 : \mathcal{A}(\Delta^\bullet) \rightrightarrows \mathcal{B}(\Delta^\bullet)$$

between two simplicial abelian groups give the same morphisms on the homology of the associated Moore complexes.

Lemma 4.4. *Let $\varphi_0, \varphi_1, \varphi_2 : \mathcal{A} \rightarrow \mathcal{B}$ be morphisms of presheaves of abelian groups and let $\varphi_0 \xrightarrow{H'} \varphi_1$ and $\varphi_1 \xrightarrow{H''} \varphi_2$. Then*

$$\varphi_0 \xrightarrow{H' + H'' - H_{\varphi_1}} \varphi_2$$

Lemma 4.5. *Let \mathcal{A} and \mathcal{B} be two presheaves of abelian groups on the category of k -smooth schemes and let $\varphi_0 \xrightarrow{H} \varphi_1$. Let $\rho : \mathcal{A}' \rightarrow \mathcal{A}$ be a morphism. Then $\varphi_0 \circ \rho \xrightarrow{H \circ \Psi} \varphi_1 \circ \rho$. Moreover, let $\eta : \mathcal{B} \rightarrow \mathcal{B}'$ be a morphism, then $\psi \circ \varphi_0 \xrightarrow{\Psi \circ H} \psi \circ \varphi_1$ with $\Psi = \underline{\text{Hom}}(\mathbb{A}^1, \psi) : \underline{\text{Hom}}(\mathbb{A}^1, \mathcal{B}) \rightarrow \underline{\text{Hom}}(\mathbb{A}^1, \mathcal{B}')$.*

We now want to discuss matrices actions on framed correspondences and associated homotopies. Let X and Y be k -smooth schemes and $A \in GL_n(k)$ be a matrix. Then A defines an automorphism

$$\varphi_A : \text{Fr}_n(- \times X, Y) \rightarrow \text{Fr}_n(- \times X, Y)$$

of the presheaf $\text{Fr}_n(- \times X, Y)$ in the following way. Given $W \in \text{Sm}/k$ and $a = ((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) \in \text{Fr}_n(W \times X, Y)$, set

$$\begin{aligned} \varphi_A((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) \\ := ((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, A \circ (\varphi_1, \varphi_2, \dots, \varphi_n), g), \end{aligned}$$

where A is regarded as a linear automorphism of \mathbb{A}_k^n .

The automorphism φ_A of the presheaf $\text{Fr}_n(- \times X, Y)$ induces an automorphism of the free abelian presheaf $\mathbb{Z}[\text{Fr}_n(- \times X, Y)]$ and an automorphism φ_A of the the presheaf of abelian groups $\mathbb{Z}\text{Fr}_n(- \times X, Y)$.

Definition 4.6. Let $A \in SL_n(k)$. Choose a matrix $A_s \in SL_n(k[s])$ such that $A_0 = id$ and $A_1 = A$. The matrix A_s , regarded as a morphism $\mathbb{A}^n \times \mathbb{A}^1 \rightarrow \mathbb{A}^n$, gives rise to an \mathbb{A}^1 -homotopy h between id and φ_A as follows. Given $a = (\alpha, f, Z, U, \varphi, g) = ((\alpha_1, \alpha_2, \dots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \dots, \varphi_n), g) \in \text{Fr}_n(W \times X, Y)$, one sets

$$h(a) = (\alpha, f \times id_{\mathbb{A}^1}, Z \times \mathbb{A}^1, U \times \mathbb{A}^1, A_s \circ (\varphi \times id_{\mathbb{A}^1}), g \circ pr_U) \in \text{Fr}_n(W \times X \times \mathbb{A}^1, Y).$$

Clearly, $h_0(a) = a$ and $h_1(a) = \varphi_A(a)$. By linearity the homotopy h induces an \mathbb{A}^1 -homotopy H_{A_s}

$$id \xrightarrow{H_{A_s}} \varphi_A : \mathbb{Z}\text{Fr}_n(- \times X, Y) \rightrightarrows \mathbb{Z}\text{Fr}_n(- \times X, Y)$$

between the identity id and the morphism φ_A .

Lemma 4.7. Let $\rho : \mathbb{ZF}_m(- \times X, Y) \rightarrow \mathbb{ZF}_n(- \times X, Y)$ be a presheaf morphism. Let $A \in SL_n(k)$, $A_s \in SL_n(k[s])$ and H_{A_s} be as in Definition 4.6. Then one has

$$\rho \xrightarrow{H_{A_s} \circ \rho} \varphi_A \circ \rho : \mathbb{ZF}_m(- \times X, Y) \rightrightarrows \mathbb{ZF}_n(- \times X, Y).$$

For $b \in \mathbb{ZF}_m(Y, S)$ define a presheaf morphism

$$\varphi_b : \mathbb{ZF}_n(- \times X, Y) \rightarrow \mathbb{ZF}_{n+m}(- \times X, S)$$

sending $a \in \mathbb{ZF}_n(W \times X, Y)$ to $b \circ a \in \mathbb{ZF}_{n+m}(- \times X, S)$. Also, any $b \in \mathbb{ZF}_m(pt, pt)$ defines a morphism of presheaves

$$- \boxtimes b : \mathbb{ZF}_n(- \times X, Y) \rightarrow \mathbb{ZF}_{n+m}(- \times X, Y)$$

sending $a \in \mathbb{ZF}_n(W \times X, Y)$ to $a \boxtimes b \in \mathbb{ZF}_{n+m}(- \times X, Y)$.

The next three lemmas are straightforward.

Lemma 4.8. Let $b_1, b_2 \in \mathbb{ZF}_m(Y, S)$ be such that $b_1 \sim b_2$, then

$$\varphi_{b_1} \sim \varphi_{b_2} : \mathbb{ZF}_n(- \times X, Y) \rightrightarrows \mathbb{ZF}_{n+m}(- \times X, S).$$

Lemma 4.9. Let $b_1, b_2 \in \mathbb{ZF}_m(pt, pt)$ and $h \in \mathbb{ZF}_m(\mathbb{A}^1, pt)$ be such that $b_1 \stackrel{h}{\sim} b_2$, then

$$(- \boxtimes b_1) \xrightarrow{- \boxtimes h} (- \boxtimes b_2) : \mathbb{ZF}_n(- \times X, Y) \rightrightarrows \mathbb{ZF}_{n+m}(- \times X, Y).$$

Lemma 4.10. Let $z \in \mathbb{A}^m$ be a k -rational point. Set $U' = (\mathbb{A}^m)_z^h$ to be the henselization of \mathbb{A}^m at the point z . Let $i_z : pt \hookrightarrow U'$ be the closed point of U' . Let $U'_s := (\mathbb{A}^1 \times \mathbb{A}^N)_{\mathbb{A}^1 \times z}^h$ be the henselization of $\mathbb{A}^1 \times \mathbb{A}^m$ at $\mathbb{A}^1 \times z$. Then there is a morphism $H_s : U'_s \rightarrow U'$ such that:

- (a) $H_1 : U' \rightarrow U'$ is the identity morphism;
- (b) $H_0 : U' \rightarrow U'$ coincides with the composite morphism $U' \xrightarrow{p} pt \xrightarrow{i_z} U'$, where $p : U' \rightarrow pt = Spec(k)$ is the structure morphism.

The preceding lemma implies the following

Corollary 4.11. Let $z \in \mathbb{A}^m$ be a k -rational point. Let $h_s = (\mathbb{A}^1 \times z, U'_s, \psi; H_s) \in Fr_N(\mathbb{A}^1, U')$. Then one has:

- (a) $h_1 = (z, U', \psi; id_{U'}) \in Fr_N(pt, U')$;
- (b) $h_0 = (z, U', \psi; i_z \circ p) = i_z \circ (\{0\}, U', \psi; p) \in Fr_N(pt, U')$, where $p : U' \rightarrow pt = Spec(k)$ is the structure morphism and $i_z : pt \hookrightarrow U'$ is the closed point of U' .

Lemma 4.12. Let $z \in \mathbb{A}^m$ be a k -rational point. Let Y be a k -smooth scheme and let $(z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g) \in Fr_m(pt, Y)$ be a framed correspondence. Then

$$(z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g) \sim (z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), c_{g(0)}),$$

where $c_{g(0)} = g(0) \circ p : U \xrightarrow{p} pt \xrightarrow{g(0)} Y$.

Proof. Let U', U'_s, i_z and h_s be as in Corollary 4.11. Let $\pi : U' \rightarrow U$ be the canonical morphism. Take $h_s = (\mathbb{A}^1 \times z, U'_s, \varphi \circ \pi, H_s) \in Fr_m(\mathbb{A}^1, U')$ and $h'_s = g \circ \pi \circ h'_s \in Fr_m(pt, Y)$. We want to check that $h'_1 = (z, U, \varphi, g)$ and $h'_0 = (z, U, \varphi, c_{g(0)})$. This will prove our statement. One has,

$$\begin{aligned} h'_1 &= (g \circ \pi) \circ h_1 = (g \circ \pi) \circ (z, U', \varphi \circ \pi; id_{U'}) = (z, U', \varphi \circ \pi; g \circ \pi) = (z, U, \varphi; g), \\ h'_0 &= (g \circ \pi) \circ h_0 = (g \circ \pi) \circ (z, U', \varphi \circ \pi; i_z \circ p) = (z, U', \varphi \circ \pi; g \circ \pi \circ i_z \circ p) = \end{aligned}$$

$$= (z, U', \varphi \circ \pi; c_{g(0)} \circ \pi) = (z, U, \varphi; c_{g(0)})$$

as required. \square

Lemma 4.13. *Let Y be a k -smooth scheme and let $(Z, U, \varphi, g) \in \text{Fr}_1(\text{pt}, Y)$ be a framed correspondence. Suppose that $U \subset \mathbb{A}^1$ and $\varphi = p(t) \in k[t]$ is a polynomial, where t is the coordinate function on \mathbb{A}^1 .*

(1) *Then for every $a \in k$ we have*

$$(Z, U, p(t), g(t)) \sim (m_a^{-1}(Z), m_a^{-1}(U), p(t-a), g(t-a)) \in \text{Fr}_1(\text{pt}, Y),$$

where $m_a: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is given by $m_a(t) = t - a$.

(2) *If $Z = \{x_0\}$ for some $x_0 \in k$ and $p(t) = (t - x_0)^n r(t)$, $r(x_0) \neq 0$, and $r(t)$ is invertible on U , then*

$$(Z, U, p(t), g) \sim (\{0\}, \mathbb{A}^1, r(x_0)t^n, c_{g(x_0)}) \in \text{Fr}_1(\text{pt}, Y),$$

where $c_{g(x_0)}: \mathbb{A}^1 \rightarrow \text{pt} \xrightarrow{g(x_0)} Y$ is the constant map taking \mathbb{A}^1 to the point $g(x_0) \in Y$.

Proof. (1) The homotopy is given by

$$(m_{sa}^{-1}(Z), m_{sa}^{-1}(U), p(t-sa), g(t-sa)) \in \text{Fr}_1(\mathbb{A}^1, Y),$$

where s is the homotopy parameter and $m_{sa}: \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the morphism $m_{sa}(t) = t - sa$.

(2) Using the preceding statement, we may assume that $x_0 = 0$. Consider a polynomial

$$h(s, t) = sr(t)t^n + (1-s)r(0)t^n \in k[t, s].$$

One easily sees that $Z(h) = (0 \times \mathbb{A}^1) \sqcup S$. The framed correspondence

$$(\{0\} \times \mathbb{A}^1, (U \times \mathbb{A}^1) \setminus S, sr(t)t^n + (1-s)r(0)t^n, g) \in \text{Fr}_1(\mathbb{A}^1, Y)$$

yields the relation $(\{0\}, U, r(t)t^n, g) \sim (\{0\}, U, r(0)t^n, g)$ in $\text{Fr}_1(\text{pt}, Y)$. Lemma 4.12 shows that

$$(\{0\}, U, r(0)t^n, g) \sim (\{0\}, U, r(0)t^n, g(0)) = (\{0\}, \mathbb{A}^1, r(0)t^n, g(0)) \in \text{Fr}_1(\text{pt}, Y)$$

and our lemma follows. \square

Lemma 4.14. *Let $(Z, \mathbb{A}^1, p(t), c) \in \text{Fr}_1(\text{pt}, \text{pt})$ be a framed correspondence, where $p(t) = at^n + \dots$ is a polynomial of degree n with the leading coefficient a and $c: \mathbb{A}^1 \rightarrow \text{pt}$ is the canonical projection. Then*

$$(Z, \mathbb{A}^1, p(t), c) \sim (\{0\}, \mathbb{A}^1, at^n, c) \in \text{Fr}_1(\text{pt}, \text{pt}).$$

Proof. The homotopy is given by the framed cycle

$$(Z(p(t) + s(at^n - p(t))), \mathbb{A}^1 \times \mathbb{A}^1, p(t) + s(at^n - p(t)), c'),$$

where s is the homotopy parameter and $c': \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \text{pt}$ is the canonical projection. \square

5. HOMOTOPIES FOR COORDINATES SWAP OF $\mathbb{G}_m \times \mathbb{G}_m$

Denote $\varepsilon = (\{0\}, \mathbb{A}^1, -t, c) \in \text{Fr}_1(\text{pt}, \text{pt})$, where $c: \mathbb{A}^1 \rightarrow \text{pt}$ is the canonical projection.

Proposition 5.1. *Let Y be a k -smooth scheme. Then the canonical homomorphism*

$$H_0(\mathbb{Z}\mathbb{F}(\Delta^\bullet \times \mathbb{G}_m \times \mathbb{G}_m, Y)) \rightarrow H_0(\mathbb{Z}\mathbb{F}(\Delta_{\text{Spec } k(t, u)}^\bullet, Y))$$

is injective.

Proof. By [GP2, 2.15(1)] the canonical homomorphisms

$$H_0(\mathbb{Z}F(\Delta^\bullet \times \mathbb{G}_m \times \mathbb{G}_m, Y)) \rightarrow H_0(\mathbb{Z}F(\Delta^\bullet \times \mathbb{G}_{m,k(u)}, Y))$$

and

$$H_0(\mathbb{Z}F(\Delta^\bullet \times \mathbb{G}_{m,k(u)}, Y)) \rightarrow H_0(\mathbb{Z}F(\Delta_{\text{Spec } k(t,u)}^\bullet, Y))$$

are injective, hence the lemma. \square

Lemma 5.2. *Let F/k be a field, choose $x, y \in F^\times$ such that $x \neq y^{\pm 1}$ and let u_1, u_2 be coordinates on $\mathbb{G}_m \times \mathbb{G}_m$. Consider morphisms $f, g: \text{Spec } F \rightarrow \mathbb{G}_m \times \mathbb{G}_m$ given by $u_1 \mapsto x, u_2 \mapsto y$ and $u_1 \mapsto y, u_2 \mapsto x$ respectively. Then for $p = (\text{id} - e_1) \boxtimes (\text{id} - e_1)$ we have $p \circ f \sim p \circ (-\varepsilon \boxtimes g)$ in $\mathbb{Z}F(\text{Spec } F, \mathbb{G}_m \times \mathbb{G}_m)$.*

Proof. The adjunction isomorphism

$$\mathbb{Z}F(k)(\text{Spec } F, \mathbb{G}_m \times \mathbb{G}_m) \cong \mathbb{Z}F(F)(\text{Spec } F, \mathbb{G}_{m,F} \times \mathbb{G}_{m,F})$$

implies it is sufficient to verify the case $F = k$. So we have morphisms $f, g: \text{pt} \rightarrow \mathbb{G}_m, \text{pt} \mapsto (x, y)$ and $\text{pt} \mapsto (y, x)$ respectively. Taking suspensions, we obtain framed correspondences

$$(\{0\}, \mathbb{A}^1, t, c_{(x,y)}), (\{0\}, \mathbb{A}^1, t, c_{(y,x)}) \in \text{Fr}_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m),$$

where $c_{(x,y)}$ and $c_{(y,x)}$ are morphisms on \mathbb{A}^1 sending it to the points (x, y) and (y, x) respectively.

Consider $h(s, t) = \frac{1}{x-y}(t^2 - (s(x+y) + (1-s)(xy+1))t + xy) \in k[s, t, t^{-1}] = k[\mathbb{A}^1 \times \mathbb{G}_m]$ and a framed correspondence

$$H_s := (Z(h), \mathbb{A}^1 \times \mathbb{G}_m, h(s, t), (t, xyt^{-1})) \in \text{Fr}_1(\mathbb{A}^1, \mathbb{G}_m \times \mathbb{G}_m). \quad (4)$$

We have $h(0, t) = \frac{1}{x-y}(t - xy)(t - 1)$ and $h(1, t) = \frac{1}{x-y}(t - x)(t - y)$. Using the additivity property for supports in $\mathbb{Z}F_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m)$ (see Definition 2.5) and Lemma 4.13 we will check below that

$$(\{0\}, \mathbb{A}^1, t, c_{(x,y)}) + (\{0\}, \mathbb{A}^1, -t, c_{(y,x)}) \sim (\{0\}, \mathbb{A}^1, \frac{1-xy}{x-y}t, c_{(1,xy)}) + (\{0\}, \mathbb{A}^1, \frac{xy-1}{x-y}t, c_{(xy,1)}) \quad (5)$$

in $\mathbb{Z}F_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m)$. The composition with the idempotent p annihilates all extra summands and proves the lemma.

In order to prove the relation (5), consider the frame correspondence (4) in $\mathbb{Z}F_1(\mathbb{A}^1, \mathbb{G}_m \times \mathbb{G}_m)$. Observe that in $\mathbb{Z}F_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m)$

$$\begin{aligned} H_1 &= (Z(h(t, 1), \mathbb{G}_m, h(1, t), (t, xyt^{-1}))) = \\ &= (\{x\}, \mathbb{G}_m - \{y\}, \frac{1}{x-y}(t-x)(t-y), (t, xyt^{-1})) + (\{y\}, \mathbb{G}_m - \{x\}, \frac{1}{x-y}(t-x)(t-y), (t, xyt^{-1})). \end{aligned}$$

By Lemma 4.13 one has in $\mathbb{Z}F_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m)$

$$\begin{aligned} (\{x\}, \mathbb{G}_m - \{y\}, \frac{1}{x-y}(t-x)(t-y), (t, xyt^{-1})) &\sim (\{0\}, \mathbb{A}^1, \frac{x-y}{x-y}t, c_{(x,y)}) = (\{0\}, \mathbb{A}^1, t, c_{(x,y)}), \\ (\{y\}, \mathbb{G}_m - \{x\}, \frac{1}{x-y}(t-x)(t-y), (t, xyt^{-1})) &\sim (\{0\}, \mathbb{A}^1, \frac{y-x}{x-y}t, c_{(x,y)}) = (\{0\}, \mathbb{A}^1, -t, c_{(x,y)}). \end{aligned}$$

Thus $H_1 \sim (\{0\}, \mathbb{A}^1, t, c_{(x,y)}) + (\{0\}, \mathbb{A}^1, -t, c_{(y,x)})$ in $\mathbb{Z}F_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m)$. Similar computations show that $H_0 \sim (\{0\}, \mathbb{A}^1, \frac{1-xy}{x-y}t, c_{(1,xy)}) + (\{0\}, \mathbb{A}^1, \frac{xy-1}{x-y}t, c_{(xy,1)})$ in $\mathbb{Z}F_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m)$. The relation (5) is proved, and hence the lemma. \square

Proposition 5.3. *Let $\tau: \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m \times \mathbb{G}_m$ be the permutation of coordinates morphism. Denote $p = (\text{id} - e_1) \boxtimes (\text{id} - e_1)$. Then $p \circ \text{id} \sim p \circ (-\varepsilon \boxtimes \tau)$ in $\mathbb{Z}\mathbb{F}(\mathbb{G}_m \times \mathbb{G}_m, \mathbb{G}_m \times \mathbb{G}_m)$.*

Proof. Let u_1 and u_2 be coordinate functions on $\mathbb{G}_m \times \mathbb{G}_m$. In view of Proposition 5.1 it is sufficient to show that $p \circ f = p \circ (-\varepsilon \boxtimes g)$ in $H_0(\mathbb{Z}\mathbb{F}(\Delta_{k(u_1, u_2)}^\bullet, \mathbb{G}_m \times \mathbb{G}_m))$, where $f: \text{Spec}k(u_1, u_2) \rightarrow \text{Spec}k[u_1, u_2]$ is the canonical embedding and $g: \text{Spec}k(u_1, u_2) \rightarrow \text{Spec}k[u_1, u_2]$ is given by $g^*(u_1) = u_2, g^*(u_2) = u_1$. The last assertion follows from Lemma 5.2. \square

It follows from Proposition 5.3 that there exists a homotopy $\Psi \in \mathbb{Z}\mathbb{F}_n(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, \mathbb{G}_m \times \mathbb{G}_m)$ such that $i_0^*(\Psi) = p \circ (-\varepsilon \boxtimes \Sigma^{n-1} \text{id})$ and $i_1^*(\Psi) = p \circ \Sigma^n \tau$, where $p = (\text{id} - e_1) \boxtimes (\text{id} - e_1)$.

Recall that $\Sigma = (\{0\}, \mathbb{A}^1, t) \in \mathbb{Z}\mathbb{F}_1(pt, pt)$. For every $k > 0$ we write Σ^k to denote $\Sigma \boxtimes \cdots \boxtimes \Sigma \in \mathbb{Z}\mathbb{F}_k(pt, pt)$.

Lemma 5.4. *Let X, Y be k -smooth schemes and $m \geq 0$ be an integer, and let n be the same as in the choice of the element Ψ . Let $\tau: \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m \times \mathbb{G}_m$ be the permutation of coordinates morphism. Consider two presheaf morphisms*

$$\begin{aligned} (-\boxtimes \Sigma^{2n}) : \mathbb{Z}\mathbb{F}_m(- \times X \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m) &\rightarrow \mathbb{Z}\mathbb{F}_{m+2n}(- \times X \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m), \\ (-\boxtimes \Sigma^{2n}) \circ sw : \mathbb{Z}\mathbb{F}_m(- \times X \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m) &\rightarrow \mathbb{Z}\mathbb{F}_{m+2n}(- \times X \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m), \end{aligned}$$

where $sw(a) = (\text{id}_Y \times \tau) \circ a \circ (\text{id}_X \times \tau)$. Then there is a morphism of presheaves of abelian groups

$$H : \mathbb{Z}\mathbb{F}_m(- \times X \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m) \rightarrow \mathbb{Z}\mathbb{F}_{m+2n}(- \times X \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$$

such that for any $a \in \mathbb{Z}\mathbb{F}_m(W \times X \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1))$ one has

$$a \boxtimes \Sigma^{2n} = H_0(a) \text{ and } H_1(a) = \Sigma^{2n}[(\text{id}_Y \times \tau) \circ a \circ (\text{id}_X \times \tau)].$$

Moreover, both $H_0(a)$ and $H_1(a)$ are in $\mathbb{Z}\mathbb{F}_{m+2n}(W \times X \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1) \times \mathbb{A}^1, Y \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1))$.

Proof. Given any element $a \in \mathbb{Z}\mathbb{F}_m(W \times X \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m)$, set

$$\begin{aligned} H'(a) &= (\text{id}_Y \times \Psi) \circ (a \times \text{id}_{\mathbb{A}^1}) \circ (\text{id}_Y \times \Psi \times \text{id}_{\mathbb{A}^1}) \circ (\text{id}_{X \times \mathbb{G}_m \times \mathbb{G}_m} \times \Delta) \in \\ &\in \mathbb{Z}\mathbb{F}_{m+2n}(W \times X \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m), \end{aligned}$$

where $\Delta : \mathbb{A}^1 \rightarrow \mathbb{A}^1 \times \mathbb{A}^1$ is the diagonal morphism. Then for any element $a \in \mathbb{Z}\mathbb{F}_m(W \times X \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1))$ one has

$$H'(a)_0 = [\text{id}_Y \times \Sigma^{n-1}(\varepsilon)] \circ a \circ [\text{id}_X \times \Sigma^{n-1}(\varepsilon)] \text{ and } H'(a)_1 = [\text{id}_Y \times \Sigma^n(\tau)] \circ a \circ [\text{id}_X \times \Sigma^n(\tau)].$$

It is easy to see that there are matrices $A, B \in SL_{m+2n}(k)$ such that for any element a in $\mathbb{Z}\mathbb{F}_m(W \times X \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1))$ one has

$$\begin{aligned} \varphi_A([\text{id}_Y \times \Sigma^{n-1}(\varepsilon)] \circ a \circ [\text{id}_X \times \Sigma^{n-1}(\varepsilon)]) &= a \boxtimes \Sigma^{2n} = \Sigma^{2n}(a), \\ \varphi_B([\text{id}_Y \times \Sigma^n(\tau)] \circ a \circ [\text{id}_X \times \Sigma^n(\tau)]) &= ([\text{id}_Y \times \tau] \circ a \circ [\text{id}_X \times \tau]) \boxtimes \Sigma^{2n} = \Sigma^{2n}([\text{id}_Y \times \tau] \circ a \circ [\text{id}_X \times \tau]). \end{aligned}$$

Choose matrices $A_s, B_s \in SL_{m+2n}(k[s])$ such that $A_0 = \text{id}, A_1 = A, B_0 = \text{id}, B_1 = B$. Then for the matrix $C_s = B_s \circ A_{1-s} \in SL_{m+2n}(k[s])$ one has $C_0 = A, C_1 = B$. Set $H = \varphi_{C_s} \circ H'$. Then for the chosen element $a \in \mathbb{Z}\mathbb{F}_m(W \times X \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1) \wedge (\mathbb{G}_m, 1))$, one has

$$H_0(a) = \varphi_A(H'(a)_0) = \Sigma^{2n}(a) \text{ and } H_1(a) = \varphi_B(H'(a)_1) = \Sigma^{2n}([\text{id}_Y \times \tau] \circ a \circ [\text{id}_X \times \tau]),$$

as was to be proved. \square

6. THE INVERSE MORPHISM

The main aim of this section is to define for any integers $n, m \geq 0$ a subpresheaf $\mathbb{Z}F_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ of the presheaf $\mathbb{Z}F_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ and define a morphism of abelian presheaves

$$\rho_n : \mathbb{Z}F_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) \rightarrow \mathbb{Z}F_m(-, Y).$$

We also prove certain properties of morphisms ρ_n and of presheaves $\mathbb{Z}F_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ which are used in the proof of the Linear Cancellation Theorem (Theorem D).

We begin with some general remarks. Let X and Y be k -smooth schemes. Consider a framed correspondence

$$a = (Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g) \in \text{Fr}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m).$$

Let $(U, p : U \rightarrow \mathbb{A}^m \times (X \times \mathbb{G}_m), s : Z \rightarrow U)$ be the étale neighborhood of Z in $\mathbb{A}^m \times (X \times \mathbb{G}_m)$ from the definition of the framed correspondence a . Let t be the invertible function on $X \times \mathbb{G}_m$ corresponding to the projection on \mathbb{G}_m and u be invertible function on $Y \times \mathbb{G}_m$ corresponding to the projection on \mathbb{G}_m . Let $f_2 = g^*(u)$ and $f_1 = p_{X \times \mathbb{G}_m}^*(t)$ be two invertible functions on U , where $p_{X \times \mathbb{G}_m} = pr_{X \times \mathbb{G}_m} \circ p : U \rightarrow X \times \mathbb{G}_m$. Set $g = (g_1, g_2)$, where $g_1 = pr_Y \circ g$ and $g_2 = pr_{\mathbb{G}_m} \circ g$.

Since Z is finite over $X \times \mathbb{G}_m$, the $\mathcal{O}_{X \times \mathbb{G}_m \times Y \times \mathbb{G}_m}$ -sheaf $P_a := \mathcal{O}_U / (\varphi_1, \varphi_2, \dots, \varphi_m)$ is finite over $X \times \mathbb{G}_m$. Since the sheaf P_a is finite over $X \times \mathbb{G}_m$, it is automatically flat over $X \times \mathbb{G}_m$.

Let Z_n^+ be the closed subset of Z defined by the equation $(f_1^{n+1} - 1)|_Z = 0$. Let Z_n^- be the closed subset of Z defined by the equation $(f_1^{n+1} - f_2)|_Z = 0$. Note that Z_n^+ is finite over X if and only if $\mathcal{O}_U / (f_1^{n+1} - 1, \varphi_1, \varphi_2, \dots, \varphi_m)$ is finite over X . By [S, 4.1] the latter \mathcal{O}_X -module is always finite and even flat. Note that Z_n^- is finite over X if and only if $\mathcal{O}_U / (f_1^{n+1} - f_2, \varphi_1, \varphi_2, \dots, \varphi_m)$ is finite over X . As it was mentioned above, the \mathcal{O}_X -module $P_a = \mathcal{O}_U / (\varphi_1, \varphi_2, \dots, \varphi_m)$ is finite and flat over X . By [S, 4.1] the \mathcal{O}_X -module $\mathcal{O}_U / (f_1^{n+1} - f_2, \varphi_1, \varphi_2, \dots, \varphi_m)$ is finite and even flat over X for sufficiently large n . In particular, Z_n^- is finite over X for sufficiently large n .

Definition 6.1. Let X and Y be k -smooth schemes. Consider a framed correspondence $a = (Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g) \in \text{Fr}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Set

$$\rho_{n,fr}^+(a) := (Z_n^+, U, (f_1^{n+1} - 1, \varphi_1, \varphi_2, \dots, \varphi_m), g_1)$$

and

$$\rho_{n,fr}^-(a) := (Z_n^-, U, (f_1^{n+1} - f_2, \varphi_1, \varphi_2, \dots, \varphi_m), g_1).$$

As we have mentioned above, Z_n^+ is finite over X for all $n \geq 0$, hence $\rho_{n,fr}^+(a) \in \mathbb{Z}F_{m+1}(X, Y)$. We say that $\rho_{n,fr}^-(a)$ is *defined* if Z_n^- is finite over X , which is equivalent to saying that the \mathcal{O}_X -module $P_a / (f_1^{n+1} - f_2)P_a$ is finite and flat over X . If $\rho_{n,fr}^-(a)$ is defined, then we set

$$\rho_{n,fr}(a) = \rho_{n,fr}^+(a) - \rho_{n,fr}^-(a) \in \mathbb{Z}F_{m+1}(X, Y)$$

and say that $\rho_{n,fr}(a)$ is *defined*.

Given integers $m, n \geq 0$, denote by $\text{Fr}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ the subset of those framed correspondences $a \in \text{Fr}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ for which the \mathcal{O}_X -module $P_a / (f_1^{n+1} - f_2)P_a$ is finite over X (that is $\rho_{n,fr}(a)$ is defined). It follows from [S, 4.4] that the assignment $X' \mapsto \text{Fr}_m^{(n)}(X' \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a subpresheaf of $\text{Fr}_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$.

Definition 6.2. Define a presheaf of abelian groups $\mathbb{Z}F_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ as follows. Its sections on X is the abelian group $\mathbb{Z}[\text{Fr}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)]$ modulo a subgroup generated by all elements of the form

$$(Z_1 \sqcup Z_2, U_1 \sqcup U_2, \varphi_1 \sqcup \varphi_2, g_1 \sqcup g_2) - (Z_1, U_1, \varphi_1, g_1) - (Z_2, U_2, \varphi_2, g_2).$$

It is straightforward to check that $\mathbb{Z}F_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a free abelian group with a free basis consisting of the elements of the form $a = (Z, U, \varphi, g)$, where Z is connected and the \mathcal{O}_X -module $P_a/(f_1^{n+1} - f_2)P_a$ is finite and flat over X . Moreover, the group $\mathbb{Z}F_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a subgroup of the group $\mathbb{Z}F_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$, and $\mathbb{Z}F_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a subpresheaf of the presheaf $\mathbb{Z}F_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$.

It follows from [S, 4.4] that for any morphism $f : X' \rightarrow X$ of smooth varieties the following diagram is commutative

$$\begin{array}{ccc} \text{Fr}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) & \xrightarrow{(f \times \text{id})^*} & \text{Fr}_m^{(n)}(X' \times \mathbb{G}_m, Y \times \mathbb{G}_m) \\ \downarrow \rho_{n,fr} & & \downarrow \rho_{n,fr} \\ \mathbb{Z}F_{m+1}(X, Y) & \xrightarrow{f^*} & \mathbb{Z}F_{m+1}(X', Y). \end{array}$$

We see that $\rho_{n,fr} : \text{Fr}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) \rightarrow \mathbb{Z}F_{m+1}(-, Y)$ is a morphism of pointed presheaves. We can extend it to get a morphism of presheaves of abelian groups $\mathbb{Z}[\text{Fr}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)] \rightarrow \mathbb{Z}F_{m+1}(-, Y)$. This morphism annihilates the elements of the form

$$(Z_1 \sqcup Z_2, U_1 \sqcup U_2, \varphi_1 \sqcup \varphi_2, g_1 \sqcup g_2) - (Z_1, U_1, \varphi_1, g_1) - (Z_2, U_2, \varphi_2, g_2).$$

Definition 6.3. The above arguments show that the presheaf morphism $\rho_{n,fr}$ induces a unique presheaf of abelian groups morphism

$$\rho_n : \mathbb{Z}F_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) \rightarrow \mathbb{Z}F_{m+1}(-, Y)$$

such that for any $a \in \mathbb{Z}F_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ one has $\rho_n(a) = \rho_{n,fr}(a)$. We also call ρ_n the *inverse morphism*.

Lemma 6.4. *The following relations are true:*

$$\text{Fr}_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) = \text{colim}_n \text{Fr}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m),$$

$$\mathbb{Z}F_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) = \text{colim}_n \mathbb{Z}F_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m).$$

This lemma follows from the following

Proposition 6.5. ([S, 4.1]) *For any framed correspondence $a \in \text{Fr}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ one has:*

- (a) *for any $n = 0$, the sheaf $P_a/(f_1^{n+1} - 1)P_a$ is finite and flat over X ;*
- (b) *there exists an integer N such that, for any $n \geq N$, the sheaf $P_a/(f_1^{n+1} - f_2)P_a$ is finite and flat over X .*

We shall need the following obvious property of ρ_n .

Lemma 6.6. For any integers $m, n, r \geq 0$, the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}F_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) & \xrightarrow{\Sigma^r} & \mathbb{Z}F_{m+r}^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) \\ \downarrow \rho_n & & \downarrow \rho_n \\ \mathbb{Z}F_{m+1}(-, Y) & \xrightarrow{\Sigma^r} & \mathbb{Z}F_{m+1+r}(-, Y). \end{array}$$

Lemma 6.7. Let X and Y be k -smooth schemes. Then for any integers m and n and any $a \in \mathbb{Z}F_m(X, Y)$, one has $a \boxtimes (\text{id} - e_1) \in \mathbb{Z}F_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. In particular, for any integers m and n there is defined the composite morphism

$$\rho_n \circ (- \boxtimes (\text{id} - e_1)) : \mathbb{Z}F_m(- \times X, Y) \rightarrow \mathbb{Z}F_m^{(n)}(- \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \rightarrow \mathbb{Z}F_{m+1}(- \times X, Y).$$

Moreover, for an element $a \in \mathbb{Z}F_m(W \times X, Y)$ of the form $(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g)$ one has

$$\begin{aligned} \rho_n(a \boxtimes (\text{id} - e_1)) &= -(Z \times Z(t^{n+1} - t)), U \times \mathbb{G}_m, (t^{n+1} - t, \varphi_1, \varphi_2, \dots, \varphi_m), g) + \\ &+ (Z \times Z(t^{n+1} - 1)), U \times \mathbb{G}_m, (t^{n+1} - 1, \varphi_1, \varphi_2, \dots, \varphi_m), g) \in \mathbb{Z}F_{m+1}(W \times X, Y). \end{aligned}$$

Proof. Let $a \in \mathbb{Z}F_m(W \times X, Y)$ be the image of $(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g) \in \text{Fr}_m(W \times X, Y)$. Then

$$\begin{aligned} a \boxtimes (\text{id} - e_1) &= (Z \times \mathbb{G}_m, U \times \mathbb{G}_m, (\varphi_1, \varphi_2, \dots, \varphi_m), (g, t)) - \\ &- (Z \times \mathbb{G}_m, U \times \mathbb{G}_m, (\varphi_1, \varphi_2, \dots, \varphi_m), (g, e_1)) \in \mathbb{Z}F_m(W \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m), \end{aligned}$$

where t is the coordinate function on \mathbb{G}_m . Clearly, $Z_n^+ = Z \times Z(t^{n+1} - 1) \subset Z \times \mathbb{G}_m$ and $Z_n^- = Z \times Z(t^{n+1} - t) \subset Z \times \mathbb{G}_m$. Both sets are finite over X . Hence $a \boxtimes (\text{id} - e_1) \in \mathbb{Z}F_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ in this case. Any element of $\mathbb{Z}F_m(W \times X, Y)$ is a linear combination of elements of the form $(Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g)$. This proves the first assertion of the lemma.

Computing $\rho_n(a \boxtimes (\text{id} - e_1))$ for $a = (Z, U, (\varphi_1, \varphi_2, \dots, \varphi_m), g)$ we obtain

$$\begin{aligned} \rho_n(a \boxtimes (\text{id} - e_1)) &= -(Z \times Z(t^{n+1} - t)), U \times \mathbb{G}_m, (t^{n+1} - t, \varphi_1, \varphi_2, \dots, \varphi_m), g) + \\ &+ (Z \times Z(t^{n+1} - 1)), U \times \mathbb{G}_m, (t^{n+1} - 1, \varphi_1, \varphi_2, \dots, \varphi_m), g) \in \mathbb{Z}F_{m+1}(W \times X, Y), \end{aligned}$$

as was to be shown. \square

Lemma 6.8. Let X and Y be k -smooth schemes. Then for every even integer m and any n one has

$$\rho_n \circ (- \boxtimes (\text{id} - e_1)) \sim (- \boxtimes \varepsilon) : \mathbb{Z}F_m(- \times X, Y) \rightrightarrows \mathbb{Z}F_{m+1}(- \times X, Y),$$

where $\varepsilon = (\{0\}, \mathbb{A}^1, -t, c') \in \mathbb{Z}F_1(\text{pt}, \text{pt})$.

Proof. Set $\eta_n = \rho_n \circ (- \boxtimes (\text{id} - e_1))$. Take the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in SL_{m+1}(k)$$

and let $A_s \in SL_{m+1}(k[s])$ be such that $A_0 = \text{id}$, $A_1 = A$. Let H_{A_s} be the \mathbb{A}^1 -homotopy from Definition 4.6 between the identity and φ_A . By Definition 4.6 one has

$$\eta_n = \rho_n \circ (- \boxtimes (\text{id} - e_1)) \xrightarrow{H_{A_s} \circ \eta_n} \varphi_A \circ \rho_n \circ (- \boxtimes (\text{id} - e_1)) = \varphi_A \circ \eta_n.$$

Set $H' = H_{A_s} \circ \eta_n$. By Lemma 4.4 it remains to find an H'' such that $\varphi_A \circ \eta_n \xrightarrow{H''} (-\boxtimes \varepsilon)$ and set $H = H' + H'' - H_{\varphi_A \circ \eta_n}$. In this case by Lemma 4.4 one gets $\rho_n \circ (-\boxtimes (\text{id} - e_1)) = \eta_n \xrightarrow{H} (-\boxtimes \varepsilon)$.

To construct H'' , note that by the last statement of Lemma 6.7 one has

$$\varphi_A \circ \eta_n = -\boxtimes [Z(t^{n+1} - 1), \mathbb{G}_m, t^{n+1} - 1, c) - (Z(t^{n+1} - t), \mathbb{G}_m, t^{n+1} - t, c)]$$

and $(-\boxtimes \varepsilon) = -\boxtimes (\{0\}, \mathbb{A}^1, -t, c')$, where where $c: \mathbb{G}_m \rightarrow \text{pt}$ is the canonical projection. By Lemma 4.9 one can take H'' to be an \mathbb{A}^1 -homotopy of the form $H'' = (-\boxtimes h'')$, where $h'' \in \mathbb{Z}F_1(\mathbb{A}^1, pt)$ is such that

$$(Z(t^{n+1} - 1), \mathbb{G}_m, t^{n+1} - 1, c) - (Z(t^{n+1} - t), \mathbb{G}_m, t^{n+1} - t, c) = h''_0$$

and

$$h''_1 = (\{0\}, \mathbb{A}^1, -t, c') \in \mathbb{Z}F_1(\text{pt}, \text{pt}),$$

where $c': \mathbb{A}^1 \rightarrow \text{pt}$ is the canonical projection. Now let us find the desired element h'' . Since $t^{n+1} - 1$ does not vanish at $t = 0$, we can extend the neighborhood from \mathbb{G}_m to \mathbb{A}^1 to get an equality,

$$(Z(t^{n+1} - 1), \mathbb{G}_m, t^{n+1} - 1, c) = (Z(t^{n+1} - 1), \mathbb{A}^1, t^{n+1} - 1, c') \in \mathbb{Z}F_1(pt, pt).$$

By Lemma 4.14 there is $h''' \in \mathbb{Z}F_1(\mathbb{A}^1, pt)$ such that

$$(Z(t^{n+1} - 1), \mathbb{A}^1, t^{n+1} - 1, c') = h'''_0 \text{ and } h'''_1 = (Z(t^{n+1} - t), \mathbb{A}^1, t^{n+1} - t, c') \in \mathbb{Z}F_1(pt, pt),$$

because polynomials $t^{n+1} - t$ and $t^{n+1} - 1$ have the same degree and the same leading coefficient. Using the additivity property for supports in $\mathbb{Z}F_1(pt, pt)$ and the second statement of Lemma 4.13, we can find an element $h^{iv} \in \mathbb{Z}F_1(\mathbb{A}^1, pt)$ such that

$$(Z(t^{n+1} - t), \mathbb{G}_m, t^{n+1} - t, c) = h^{iv}_0 \text{ and } h^{iv}_1 = (Z(t^{n+1} - t), \mathbb{A}^1, t^{n+1} - t, c') - (\{0\}, \mathbb{A}^1, -t, c') \in \mathbb{Z}F_1(pt, pt)$$

Set $h'' := h''' - h^{iv} \in \mathbb{Z}F_1(\mathbb{A}^1, pt)$. Then h'' is the desired element.

Set $H'' = (-\boxtimes h'')$ and $H = H' + H'' - H_{\varphi_A \circ \eta_n}$. Then H is the desired \mathbb{A}^1 -homotopy. That is

$$\rho_n \circ (-\boxtimes (\text{id} - e_1)) \xrightarrow{H} (-\boxtimes \varepsilon)$$

and our statement follows. \square

7. THEOREM D

The main purpose of this section is to prove Theorem D. We sometimes identify simplicial abelian groups with chain complexes concentrated in non-negative degrees by using the Dold-Kan correspondence.

Lemma 7.1. *Let X and Y be k -smooth schemes and $m, r, N \geq 0$ be integers. Then for any Moore cycle $a \in \mathbb{Z}F_m(\Delta^r \times X, Y)$ of the simplicial abelian group $\mathbb{Z}F_m(\Delta^\bullet \times X, Y)$, one has $a \boxtimes (\text{id} - e_1) \in \mathbb{Z}F_m^{(N)}(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Moreover, $\rho_N(a \boxtimes (\text{id} - e_1))$ is a Moore cycle. The homology classes of Moore cycles*

$$a \boxtimes \varepsilon \text{ and } \rho_N(a \boxtimes (\text{id} - e_1))$$

coincide in $\mathbb{Z}F_{m+1}(\Delta^\bullet \times X, Y)$.

Proof. The element $a \boxtimes (\text{id} - e_1)$ is in $\mathbb{Z}F_m^{(N)}(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ by Lemma 6.7. Since $\mathbb{Z}F_m^{(N)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a presheaf, then $\partial_i(a \boxtimes (\text{id} - e_1)) \in \mathbb{Z}F_m^{(N)}(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Since the morphism ρ_N is a morphism of presheaves, then

$$\partial_i(\rho_N(a \boxtimes (\text{id} - e_1))) = \rho_N(\partial_i(a \boxtimes (\text{id} - e_1))) = \rho_N(\partial_i(a) \boxtimes (\text{id} - e_1)) = 0.$$

This proves the first assertion of the lemma.

By Lemma 6.8 the morphism

$$a' \mapsto \rho_N(a' \boxtimes (\text{id}_{\mathbb{G}_m} - e_1)) : \mathbb{Z}F_m(- \times X, Y) \rightarrow \mathbb{Z}F_m^{(N)}(- \times X, Y) \rightarrow \mathbb{Z}F_{m+1}(- \times X, Y)$$

is \mathbb{A}^1 -homotopic to the morphism $a' \mapsto a' \boxtimes \varepsilon$. Thus the corresponding morphisms of the simplicial abelian groups $\mathbb{Z}F_m(\Delta^\bullet \times X, Y) \rightrightarrows \mathbb{Z}F_{m+1}(\Delta^\bullet \times X, Y)$ induce the same morphisms on homology. Hence the homology class of the Moore cycle $\rho_N(a \boxtimes (\text{id}_{\mathbb{G}_m} - e_1))$ coincides with the homology class of the Moore cycle $a \boxtimes \varepsilon$. \square

Lemma 7.2. *One has $\varepsilon \boxtimes \varepsilon \sim \Sigma^2$ in $\mathbb{Z}F_2(\text{pt}, \text{pt})$. Moreover, for any integer $r \geq 0$ one has $\varepsilon \boxtimes \varepsilon \boxtimes \Sigma^r \sim \Sigma^{2+r}$ in $\mathbb{Z}F_{2+r}(\text{pt}, \text{pt})$.*

Corollary 7.3. *Let X and Y be k -smooth schemes and $m \geq 0$ be an integer. Then,*

$$(- \boxtimes \varepsilon^2) \sim (- \boxtimes \Sigma^2) : \mathbb{Z}F_m(- \times X, Y) \rightrightarrows \mathbb{Z}F_{m+2}(- \times X, Y)$$

and

$$(- \boxtimes \varepsilon^2 \boxtimes \Sigma^r) \sim (- \boxtimes \Sigma^{2+r}) : \mathbb{Z}F_m(- \times X, Y) \rightrightarrows \mathbb{Z}F_{m+2+r}(- \times X, Y).$$

Therefore the first pair of maps produces the same maps on homology

$$H_*(\mathbb{Z}F_m(\Delta^\bullet \times X, Y)) \rightrightarrows H_*(\mathbb{Z}F_{m+2}(\Delta^\bullet \times X, Y)).$$

Similarly, the second pair of maps gives the same maps on homology

$$H_*(\mathbb{Z}F_m(\Delta^\bullet \times X, Y)) \rightrightarrows H_*(\mathbb{Z}F_{m+2+r}(\Delta^\bullet \times X, Y)).$$

Lemma 7.4. *Let X and Y be k -smooth schemes and $m \geq 0$ be an integer. Then for any integer $r \geq 0$ one has*

$$\begin{aligned} \text{Ker}[- \boxtimes (\text{id}_{\mathbb{G}_m} - e_1) : H_r(\mathbb{Z}F_m(\Delta^\bullet \times X, Y)) &\rightarrow H_r(\mathbb{Z}F_m((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))] \subseteq \\ &\subseteq \text{Ker}[- \boxtimes \Sigma^2 : H_r(\mathbb{Z}F_m(\Delta^\bullet \times X, Y)) \rightarrow H_r(\mathbb{Z}F_{m+2}(\Delta^\bullet \times X, Y))]. \end{aligned}$$

Proof. Since all complexes of the lemma are simplicial abelian groups, we may work with the associated Moore complexes. Thus, assume that

$$a \in \mathbb{Z}F_m(\Delta^r \times X, Y)$$

is a Moore cycle for which $a \boxtimes (\text{id}_{\mathbb{G}_m} - e_1)$ is a boundary, i.e., there exists $b \in \mathbb{Z}F_m((\Delta^{r+1} \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ such that $\partial_i(b) = 0$ for $i = 0, 1, \dots, r$ and $\partial_{r+1}(b) = a \boxtimes (\text{id}_{\mathbb{G}_m} - e_1)$. By Lemma 6.4 there exists an N such that $b \in \mathbb{Z}F_m^{(N)}(\Delta^{r+1} \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Since $\mathbb{Z}F_m^{(N)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a presheaf, then $\partial_i(b) \in \mathbb{Z}F_m^{(N)}(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Since ρ_N is a presheaf morphism $\mathbb{Z}F_m^{(N)}(- \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \rightarrow \mathbb{Z}F_{m+1}(- \times X, Y)$, one has $\partial_i(\rho_N(b)) = \rho_N(\partial_i(b))$. Thus,

$$\partial_i(\rho_N(b)) = \rho_N(\partial_i(b)) = 0 \text{ for } 0 \leq i \leq r,$$

$$\partial_{r+1}(\rho_N(b)) = \rho_N(\partial_{r+1}(b)) = \rho_N(a \boxtimes (\text{id}_{\mathbb{G}_m} - e_1)).$$

We see that the homology class of the Moore cycle $\rho_N(a \boxtimes (\text{id}_{\mathbb{G}_m} - e_1))$ vanishes. By Lemma 7.1 the homology class of the Moore cycle $a \boxtimes \varepsilon$ vanishes in $H_r(\mathbb{Z}F_{m+1}(\Delta^\bullet \times X, Y))$. Thus the homology class of the Moore cycle $a \boxtimes \varepsilon \boxtimes \varepsilon$ vanishes in $H_r(\mathbb{Z}F_{m+2}(\Delta^\bullet \times X, Y))$. By Corollary 7.3 the homology class of $a \boxtimes \Sigma^2$ vanishes in $H_r(\mathbb{Z}F_{m+2}(\Delta^\bullet \times X, Y))$, too. \square

Lemma 7.5. *Let X and Y be k -smooth schemes and $m, r \geq 0$ be integers. Let n be the integer from Lemma 5.4. Then for any Moore cycle $a \in \mathbb{Z}F_m((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$ there exists an integer N such that the element $\rho_N(a)$ is defined and the homology class of the Moore cycle*

$$\Sigma^{2n}(\rho_N(a)) \boxtimes (id - e_1) \in \mathbb{Z}F_{m+2n+1}((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$$

coincides with the homology class of the Moore cycle $\Sigma^{2n}(a \boxtimes \varepsilon)$.

Proof. Set $a' = a \boxtimes (id - e_1)$. Let H be the \mathbb{A}^1 -homotopy from Lemma 5.4. Consider the element

$$H(a') \in \mathbb{Z}F_{m+2n}((\Delta^r \times X) \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m).$$

By Lemma 6.4 there is an integer N such that

$$a \in \mathbb{Z}F_m^{(N)}((\Delta^r \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$$

and

$$H(a') \in \mathbb{Z}F_{m+2n}^{(N)}((\Delta^r \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m).$$

Since a' is a Moore cycle and H is a presheaf morphism, the element $H(a')$ is a Moore cycle in $\mathbb{Z}F_{m+2n}((\Delta^\bullet \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$. Since

$$\mathbb{Z}F_{m+2n}^{(N)}((- \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$$

is a subpresheaf of $\mathbb{Z}F_m((- \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$, it follows that $H(a')$ is a Moore cycle in $\mathbb{Z}F_{m+2n}^{(N)}((\Delta^\bullet \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$.

Applying the presheaf morphism

$$\rho_N : \mathbb{Z}F_{m+2n}^{(N)}((- \times X) \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m) \rightarrow \mathbb{Z}F_{m+2n+1}((- \times X) \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m)$$

to the Moore cycle $H(a')$, we get a Moore cycle

$$\rho_N(H(a')) \in \mathbb{Z}F_{m+2n+1}((\Delta^r \times X) \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m).$$

Hence $i_0^*(\rho_N(H(a'))) \in \mathbb{Z}F_{m+2n+1}((\Delta^r \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ and $i_1^*(\rho_N(H(a'))) \in \mathbb{Z}F_{m+2n+1}((\Delta^r \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ are Moore cycles, too. Furthermore,

$$i_0^*(\rho_N(H(a'))) = \rho_N(i_0^*(H(a'))) = \rho_N(\Sigma^{2n}(a')) = \Sigma^{2n}(\rho_N(a'))$$

and

$$i_1^*(\rho_N(H(a'))) = \rho_N(i_1^*(H(a'))) = \rho_N(\Sigma^{2n}[(id_Y \times \tau) \circ a' \circ (id_X \times \tau)]).$$

The two morphisms

$$i_0^*, i_1^* : \mathbb{Z}F_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m) \rightrightarrows \mathbb{Z}F_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$$

of simplicial abelian groups induce the same morphisms on homology. The element $\rho_N(H(a'))$ is a Moore cycle. Thus the homological classes of the Moore cycles $i_0^*(\rho_N(H(a')))$ and $i_1^*(\rho_N(H(a')))$ coincide in $H_r(\mathbb{Z}F_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m))$.

By Lemma 6.6 one has $\rho_N(\Sigma^{2n}(a')) = \Sigma^{2n}(\rho_N(a'))$. Thus the first homological class is the class of $\Sigma^{2n}(\rho_N(a')) = \Sigma^{2n}(\rho_N(a \boxtimes (id - e_1)))$. By Lemma 7.1 the latter homological class coincides with the class of the element $\Sigma^{2n}(a \boxtimes \varepsilon)$.

The element $i_1^*(\rho_N(H(a')))$ coincides with $\rho_N(\Sigma^{2n}[(id_Y \times \tau) \circ (a \boxtimes (id - e_1)) \circ (id_X \times \tau)])$. By Lemma 6.6 the latter element coincides with

$$\Sigma^{2n}(\rho_N[(id_Y \times \tau) \circ (a \boxtimes (id - e_1)) \circ (id_X \times \tau)]) = \Sigma^{2n}[\rho_N(a) \boxtimes (id - e_1)].$$

Hence the homological classes $\Sigma^{2n}(a \boxtimes \varepsilon)$ and $[\Sigma^{2n}[\rho_N(a) \boxtimes (id - e_1)]]$ coincide in $H_r(\mathbb{Z}F_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m))$. Finally, the complex $\mathbb{Z}F_{n+2m+1}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge$

$(\mathbb{G}_m, 1)$ is a direct summand in $\mathbb{Z}F_{n+2m+1}((\Delta^\bullet \times X) \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ and the elements $\Sigma^{2n}(a \boxtimes \varepsilon), \Sigma^{2n}(\rho_N(a) \boxtimes (id - e_1))$ are in $\mathbb{Z}F_{n+2m+1}((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$. Hence the homological classes $[\Sigma^{2n}[\rho_N(a) \boxtimes (id - e_1)]]$ and $[\Sigma^{2n}(a \boxtimes \varepsilon)]$ coincide in $H_r(\mathbb{Z}F_{n+2m+1}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)))$. \square

Lemma 7.6. *Let X and Y be k -smooth schemes and $m, r \geq 0$ be integers. Let n be the integer from Lemma 5.4. Then*

$$\begin{aligned} Im[(- \boxtimes \Sigma^{2n+2}) : H_r(\mathbb{Z}F_m((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)) \rightarrow \\ \rightarrow H_r(\mathbb{Z}F_{m+2n+2}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)))] \subseteq \\ Im[(- \boxtimes (id_{\mathbb{G}_m} - e_1)) : H_r(\mathbb{Z}F_{m+2n+2}(\Delta^\bullet \times X, Y)) \rightarrow H_r(\mathbb{Z}F_{m+2n+2}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)))]]. \end{aligned}$$

Proof. Take a Moore cycle $a' \in \mathbb{Z}F_m((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$. Then the element $a := a' \boxtimes \varepsilon$ is a Moore cycle in $\mathbb{Z}F_{m+1}((\Delta^r \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$. By Lemma 7.5 the homology classes of $\Sigma^{2n}(a \boxtimes \varepsilon)$ and $\Sigma^{2n}(\rho_N(a)) \boxtimes (id - e_1)$ coincide in

$$H_r(\mathbb{Z}F_{m+2+2n}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))).$$

By Corollary 7.3 the homology classes of $\Sigma^{2n}(a \boxtimes \varepsilon) = \Sigma^{2n}(a' \boxtimes \varepsilon \boxtimes \varepsilon)$ and $\Sigma^{2n+2}(a')$ coincide. Hence the homology classes of $\Sigma^{2n+2}(a')$ and $\Sigma^{2n}(\rho_N(a' \boxtimes \varepsilon)) \boxtimes (id - e_1)$ coincide in $H_r(\mathbb{Z}F_{m+2+2n}((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1)))$. \square

We are now in a position to prove Theorem D.

Theorem D. *Let X and Y be k -smooth schemes. Then*

$$- \boxtimes (id_{\mathbb{G}_m} - e_1) : \mathbb{Z}F(\Delta^\bullet \times X, Y) \rightarrow \mathbb{Z}F((\Delta^\bullet \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))$$

is a quasi-isomorphism of complexes of abelian groups.

Proof. The theorem follows from Lemmas 7.4 and 7.6. \square

APPENDIX A. SOME COMMUTATIVE DIAGRAMS FOR THEOREM A

In this section we give a detailed description of some arrows which are used implicitly in the proof of Theorem A.

Let $\underline{\text{Hom}} : sPre_\bullet(Sm/k)^{op} \times Sp_{S^1}^{fr} \rightarrow Sp_{S^1}^{fr}$ be the natural functor. For a morphism $\alpha : F \rightarrow G$ in $sPre_\bullet(Sm/k)$ and $E \in Sp_{S^1}^{fr}$, set $\alpha^* := \underline{\text{Hom}}(\alpha, E)$. Given a k -smooth variety X , a morphism $f : W \rightarrow W'$ of simplicial objects in $\text{Fr}_0(k)$, and an object $F \in sPre_\bullet(k)$, we put $f_* := \underline{\text{Hom}}(F, LM_{fr}(id_X \times f))$.

We need some objects and morphisms in $sPre_\bullet(Sm/k)$. The inclusion $\mathbb{G}_m \hookrightarrow (-, \mathbb{G}_m)_+$ induces a morphism $in : (\mathbb{G}_m, 1) \rightarrow (-, \mathbb{G}_m)_+ / (-, pt)_+$ in $sPre_\bullet(Sm/k)$. Let $r : (-, \mathbb{G}_m)_+ \rightarrow (-, \mathbb{G}_m)_+ / (-, pt)_+$ be the natural projection. Let $can : (-, \mathbb{G}_m)_+ \rightarrow \mathbb{G}$ be the morphism from zero-simplices of \mathbb{G} to \mathbb{G} itself. Let $Cone \in sPre_\bullet(k)$ be the simplicial mapping cone for the identity map $(-, pt)_+ \rightarrow (-, pt)_+$. The morphism $(-, \mathbb{G}_m)_+ \rightarrow (-, pt)_+$ induces a morphism $p_{\mathbb{G}} : \mathbb{G} \rightarrow Cone$. The inclusion $(-, pt)_+ \rightarrow (-, \mathbb{G}_m)_+$ induces a morphism $i_{\mathbb{G}} : Cone \rightarrow \mathbb{G}$. Set $e_{\mathbb{G}} = i_{\mathbb{G}} \circ p_{\mathbb{G}}$. Let $can' : \mathbb{G}_m \rightarrow \mathbb{G}_m^{\wedge 1}$ be the morphism from the zero-simplices of \mathbb{G}_m to $\mathbb{G}_m^{\wedge 1}$ itself. We regard this morphism as a morphism of simplicial objects in $\text{Fr}_0(k)$. The morphism $e_{\mathbb{G}_m} : \mathbb{G}_m \xrightarrow{p} pt \xrightarrow{i} \mathbb{G}_m$ will be regarded as a morphism in $\text{Fr}_0(k)$. But sometimes it will be regarded as a morphism of $sPre_\bullet(Sm/k)$.

Define $LM_{fr}(X) \xrightarrow{-\boxtimes(id_{\mathbb{G}_m} - e_{\mathbb{G}_m})} \underline{\text{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \times \mathbb{G}_m))$ as a unique morphism such that $LM_{fr}(X) \xrightarrow{-\boxtimes(id_{\mathbb{G}_m} - e_{\mathbb{G}_m})} \underline{\text{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \times \mathbb{G}_m)) \hookrightarrow \underline{\text{Hom}}(\mathbb{G}_m, LM_{fr}(X \times \mathbb{G}_m))$ coincides with $-\boxtimes(id_{\mathbb{G}_m} - e_{\mathbb{G}_m})$. Let us define

$$\overline{can^* \circ (id_{\mathbb{G}}^* - e_{\mathbb{G}}^*)} : \underline{\text{Hom}}(\mathbb{G}, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})) \rightarrow \underline{\text{Hom}}((- , \mathbb{G}_m)_+ / (- , pt)_+, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1}))$$

as a unique morphism such that $r^* \circ \overline{can^* \circ (id_{\mathbb{G}}^* - e_{\mathbb{G}}^*)} = can^* \circ (id_{\mathbb{G}}^* - e_{\mathbb{G}}^*)$.

The following lemma is useful for the proof of Theorem A.

Lemma A.1. *The composite morphism in $Sp_{S^1}^{fr}(k)$*

$$LM_{fr}(X) \xrightarrow{c_0} \underline{\text{Hom}}(\mathbb{G}, LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})) \xrightarrow{in^* \circ \overline{can^* \circ (id_{\mathbb{G}}^* - e_{\mathbb{G}}^*)}} \underline{\text{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \times \mathbb{G}_m^{\wedge 1}))$$

coincides with $can'_* \circ (-\boxtimes(id_{\mathbb{G}_m} - e_{\mathbb{G}_m}))$. Moreover, the morphism $in^* \circ \overline{can^* \circ (id_{\mathbb{G}}^* - e_{\mathbb{G}}^*)}$ is a sectionwise stable equivalence. Also, the following diagram in $Sp_{S^1}^{fr}(k)$ commutes

$$\begin{array}{ccc} LM_{fr}(X) & \xrightarrow{can'_* \circ (-\boxtimes(id_{\mathbb{G}_m} - e_{\mathbb{G}_m}))} & \underline{\text{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \times \mathbb{G}_m^{\wedge 1})) \\ \uparrow id & & \uparrow \underline{\text{Hom}}((\mathbb{G}_m, 1), can'_*|_{LM_{fr}(X \wedge (\mathbb{G}_m, 1))}) \\ LM_{fr}(X) & \xrightarrow{-\boxtimes(id_{\mathbb{G}_m} - e_{\mathbb{G}_m})} & \underline{\text{Hom}}((\mathbb{G}_m, 1), LM_{fr}(X \wedge (\mathbb{G}_m, 1))) \end{array}$$

and $\underline{\text{Hom}}((\mathbb{G}_m, 1), can'_*|_{LM_{fr}(X \wedge (\mathbb{G}_m, 1))})$ is a sectionwise stable equivalence. Here l_X is the natural morphism and $LM_{fr}(X \wedge (\mathbb{G}_m, 1)) := \text{Ker}[LM_{fr}(X \times \mathbb{G}_m) \xrightarrow{P_*} LM_{fr}(X \times pt)]$.

Proof. Similarly to Remark B.2 all the framed S^1 -spectra presheaves from the lemma are the Segal S^1 -spectra corresponding to certain framed presheaves of simplicial abelian groups. Namely, they correspond to $\mathbb{Z}F(\Delta^\bullet \times -, X)$, $\underline{\text{Hom}}(\mathbb{G}, \mathbb{Z}F(\Delta^\bullet \times -, X \times \mathbb{G}_m^{\wedge 1}))$, $\underline{\text{Hom}}((\mathbb{G}_m, 1), \mathbb{Z}F(\Delta^\bullet \times -, X \times \mathbb{G}_m^{\wedge 1}))$, $\underline{\text{Hom}}((\mathbb{G}_m, 1), \mathbb{Z}F(\Delta^\bullet \times -, X \wedge (\mathbb{G}_m, 1)))$ respectively. All the framed S^1 -spectra presheaves morphisms from the lemma correspond to certain morphisms between those framed presheaves of simplicial abelian groups. These easily yield an equality

$$can'_* \circ (-\boxtimes(id_{\mathbb{G}_m} - e_{\mathbb{G}_m})) = [in^* \circ \overline{can^* \circ (id_{\mathbb{G}}^* - e_{\mathbb{G}}^*)}] \circ c_0$$

and the commutativity of the diagram of the lemma.

We argue in the same fashion to prove the last assertion of the lemma. It suffices to show that for any $U \in Sm/k$ the morphism

$$can'_*|_{\mathbb{Z}F(U, X \wedge (\mathbb{G}_m, 1))} : \mathbb{Z}F(U, X \wedge (\mathbb{G}_m, 1)) \rightarrow \mathbb{Z}F(U, X \times \mathbb{G}_m^{\wedge 1})$$

is a quasi-isomorphism. The latter follows from the equalities $\mathbb{Z}F(U, Y \sqcup Y') = \mathbb{Z}F(U, Y) \oplus \mathbb{Z}F(U, Y')$. Hence $can'_*|_{LM_{fr}(U, X \wedge (\mathbb{G}_m, 1))}$ is indeed a stable equivalence.

The morphism $in^* \circ \overline{can^* \circ (id_{\mathbb{G}}^* - e_{\mathbb{G}}^*)}$ is a sectionwise stable equivalence for similar reasons. Indeed, the simplicial abelian group presheaf morphism

$$\underline{\text{Hom}}(\mathbb{G}, \mathbb{Z}F(X \times \mathbb{G}_m^{\wedge 1})) \xrightarrow{in^* \circ \overline{can^* \circ (id_{\mathbb{G}}^* - e_{\mathbb{G}}^*)}} \underline{\text{Hom}}((\mathbb{G}_m, 1), \mathbb{Z}F(X \times \mathbb{G}_m^{\wedge 1}))$$

is a quasi-isomorphism, because for any $Y \in Sm/k$ the morphism

$$\underline{\text{Hom}}(\mathbb{G}, \mathbb{Z}F(Y)) \xrightarrow{in^* \circ \overline{can^* \circ (id_{\mathbb{G}}^* - e_{\mathbb{G}}^*)}} \underline{\text{Hom}}((\mathbb{G}_m, 1), \mathbb{Z}F(Y))$$

is a simplicial presheaf quasi-isomorphism. \square

APPENDIX B. ANOTHER DEFINITION OF THE BISPECTRUM $M_{fr}^{\mathbb{G}}(X)$

In this section another definition of the (S^1, \mathbb{G}) -bispectrum $M_{fr}^{\mathbb{G}}(X)$ is given. This definition uses the functor associating Segal's S^1 -spectrum to a Γ -space. It does not use any extra simplicial machinery. The structure morphisms are described quite similarly to the morphism “ $-\boxtimes(\text{id}_{\mathbb{G}_m} - e_1)$ ” from Theorem D. So our aim is to construct morphisms

$$a_n : M_{fr}(X \times \mathbb{G}_m^{\wedge n}) \rightarrow \underline{\text{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge(n+1)}))$$

in $Sp_{S^1}^{fr}(k)$. We start with preparations.

There is a canonical pair of adjoint functors

$$\Phi : sPre_{\bullet}(Sm/k) \rightleftarrows sPre_{\bullet}^{fr}(k) : \Psi, \quad (6)$$

where Ψ is the forgetful functor and $\Phi(X) = \text{Fr}_+(-, X)$ for a k -smooth variety X (see [GP1, Section 4] for the definition of $\text{Fr}_+(Y, X)$). Let $sPre_{\bullet}(\text{Fr}_0(k))$ be the category of pointed simplicial presheaves on the category $\text{Fr}_0(k)$. The functor $\text{Fr}_+(k) \times \text{Fr}_0(k) \rightarrow \text{Fr}_+(k)$ sending a pair (X, Y) to $X \times Y$ has a left Kan extension

$$Pre_{\bullet}^{fr}(k) \times sPre_{\bullet}(\text{Fr}_0(k)) \xrightarrow{\boxtimes} sPre_{\bullet}^{fr}(k).$$

It takes a pair (X, Y) , $X, Y \in Sm/k$, to $X \boxtimes Y := \Phi(X \times Y) = \text{Fr}_+(-, X \times Y)$. In particular, $\text{Fr}(-, X) \boxtimes Y = \text{Fr}(-, X \times Y)$.

Below we use notation from Sections 2 and 3. In particular, \mathbb{G} and $\mathbb{G}_m^{\wedge 1}$ are as in Section 3. We regard $\mathbb{G}_m^{\wedge 1}$ as a simplicial object in $\text{Fr}_0(k)$. There is also a left Kan extension functor $\Phi_0 : sPre_{\bullet}(Sm/k) \rightarrow sPre_{\bullet}(\text{Fr}_0(k))$ adjoint to the forgetful functor Ψ_0 . We set $\mathbf{G} := \Phi_0(\mathbb{G})$. One has an obvious morphism

$$u_{\mathbb{G}} : \mathbf{G} \rightarrow \mathbb{G}_m^{\wedge 1}$$

in $sPre_{\bullet}(\text{Fr}_0(k))$ taking $+$ of to the empty object. The adjunction (6) specializes to an isomorphism

$$\text{Hom}_{sPre_{\bullet}(Sm/k)}(U_+ \wedge \mathbb{G}, \text{Fr}(-, Y \times \mathbb{G}_m^{\wedge 1})) = \text{Hom}_{sPre_{\bullet}^{fr}(k)}((U \boxtimes \mathbf{G})/(U \boxtimes +), \text{Fr}(-, Y) \boxtimes \mathbb{G}_m^{\wedge 1})$$

showing that

$$\underline{\text{Hom}}_{sPre_{\bullet}(Sm/k)}(\mathbb{G}, \text{Fr}(-, Y \times \mathbb{G}_m^{\wedge 1}))(U) = \text{Hom}_{sPre_{\bullet}^{fr}(k)}((U \boxtimes \mathbf{G})/(U \boxtimes +), \text{Fr}(-, Y) \boxtimes \mathbb{G}_m^{\wedge 1}) \quad (7)$$

For any k -smooth scheme Y , the presheaf $\text{Fr}(-, Y)$ on $\text{Fr}_*(k)$ is a pointed Nisnevich sheaf. It is covariantly functorial in Y with respect to the category $\text{Fr}_0(k)$. Moreover, $\text{Fr}(\emptyset, Y) = pt$. Thus we have a pointed Γ -space

$$\Gamma^{\text{op}} \ni (K, *) \mapsto \text{Fr}(U, Y \otimes K),$$

where the right hand side is regarded as a constant pointed simplicial set. Varying U in $\text{Fr}_*(k)$ we get a Nisnevich $\text{Fr}_*(k)$ -sheaf of pointed Γ -spaces with $\text{Fr}(\emptyset, Y \otimes K) = pt$.

Let X be a k -smooth scheme. Let A be a n -multisimplicial object of $\text{Fr}_0(k)$. Consider two n -multisimplicial Γ -spaces

$$(K, *) \mapsto \text{Fr}(-, (X \otimes K) \times A([r_1], \dots, [r_n]))(U),$$

$$(K, *) \mapsto \underline{\text{Hom}}_{sPre_{\bullet}(Sm/k)}(\mathbb{G}, \text{Fr}(-, (X \otimes K) \times A([r_1], \dots, [r_n]) \times \mathbb{G}_m^{\wedge 1}))(U).$$

Since $u_{\mathbb{G}}$ takes $+$ to the empty object, then for any k -smooth variety Y and any $\gamma \in \text{Fr}(U, Y)$ the morphism $\gamma \boxtimes u_{\mathbb{G}} : U \boxtimes \mathbf{G} \rightarrow \text{Fr}(-, Y) \boxtimes \mathbb{G}_m^{\wedge 1}$ in $sPre_{\bullet}^{fr}(k)$ takes $U \boxtimes +$ to the empty simplicial object. Thus there is a unique morphism

$$\overline{\gamma \boxtimes u_{\mathbb{G}}} : (U \boxtimes \mathbf{G}) / (U \boxtimes +) \rightarrow \text{Fr}(-, Y) \boxtimes \mathbb{G}_m^{\wedge 1}$$

in $sPre_{\bullet}^{fr}(k)$ which coincides with $\gamma \boxtimes u_{\mathbb{G}}$ after precomposing with the morphism $U \boxtimes \mathbf{G} \rightarrow (U \boxtimes \mathbf{G}) / (U \boxtimes +)$. Under the identification (7), the assignment

$$\gamma \mapsto ((U \boxtimes \mathbf{G}) / (U \boxtimes +) \xrightarrow{\overline{\gamma \boxtimes u_{\mathbb{G}}}} \text{Fr}((X \otimes K) \times A([r_1], \dots, [r_n])) \boxtimes \mathbb{G}_m^{\wedge 1})$$

is a morphism between two n -multisimplicial pointed Γ -spaces

$$((K, *) \mapsto \text{Fr}(-, (X \otimes K) \times A)(U)) \xrightarrow{\alpha_U} ((K, *) \mapsto \underline{\text{Hom}}_{sPre_{\bullet}(Sm/k)}(\mathbb{G}, \text{Fr}(-, (X \otimes K) \times A \times \mathbb{G}_m^{\wedge 1}))(U)).$$

Taking diagonals on both sides, we get a morphism of Γ -spaces. Furthermore, taking the associated Segal's S^1 -spectra, we get a morphism of S^1 -spectra

$$a_U : \text{Fr}(-, (X \otimes \mathbf{S}) \times A)(U) \rightarrow \underline{\text{Hom}}(\mathbb{G}, \text{Fr}(-, (X \otimes \mathbf{S}) \times A \times \mathbb{G}_m^{\wedge 1}))(U),$$

where \mathbf{S} is the simplicial sphere S^1 -spectrum. Clearly, the family $\alpha := \{\alpha_U | U \in \text{Fr}_*(k)\}$ is a morphism of presheaves of pointed Γ -spaces. Hence the family

$$a = \{a_U | U \in \text{Fr}_*(k)\} : \text{Fr}(-, (X \otimes \mathbf{S}) \times A) \rightarrow \underline{\text{Hom}}(\mathbb{G}, \text{Fr}(-, (X \otimes \mathbf{S}) \times A \times \mathbb{G}_m^{\wedge 1}))$$

is a morphism in the category $Sp_{S^1}^{fr}(k)$ of framed S^1 -spectra. Replacing “ $-$ ” with “ $\Delta^\bullet \times -$ ”, we get a morphism in the category $Sp_{S^1}^{fr}(k)$ of framed S^1 -spectra

$$a_A : M_{fr}(X \times A) \rightarrow \underline{\text{Hom}}(\mathbb{G}, M_{fr}(X \times A \times \mathbb{G}_m^{\wedge 1})).$$

Taking $A = \mathbb{G}_m^{\wedge n}$ we get a morphism

$$a_n := a_{\mathbb{G}_m^{\wedge n}} : M_{fr}(X \times \mathbb{G}_m^{\wedge n}) \rightarrow \underline{\text{Hom}}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge(n+1)})).$$

Definition B.1. The (S^1, \mathbb{G}) -bispectrum $M_{fr}^{\mathbb{G}}(X)$ is defined as

$$(M_{fr}(X), M_{fr}(X \times \mathbb{G}_m^{\wedge 1}), M_{fr}(X \times \mathbb{G}_m^{\wedge 2}), \dots)$$

together with the structure morphisms a_n -s. Similarly a (S^1, \mathbb{G}) -bispectrum $LM_{fr}^{\mathbb{G}}(X)$ is defined as

$$(LM_{fr}(X), LM_{fr}(X \times \mathbb{G}_m^{\wedge 1}), LM_{fr}(X \times \mathbb{G}_m^{\wedge 2}), \dots)$$

together with similar structure morphisms c_n -s. Namely, one can use for this the Γ -spaces $(K, *) \mapsto \mathbb{Z}\text{F}(U, Y \otimes K)$.

The interested reader can easily verify that the maps a_n -s and c_n -s defined above coincide with the maps (2) of Section 3.

We finish the paper by the following remark.

Remark B.2. As it is explained in Section 3 the framed S^1 -spectrum $LM_{fr}(X \times \mathbb{G}_m^{\wedge n})$ is the Eilenberg–Mac Lane spectrum associated with the complex $C_* \mathbb{Z}\text{F}(-, X \times \mathbb{G}_m^{\wedge n})$. It is easy to see that the morphism c_n is a morphism of EM-spectra associated with the simplicial abelian group presheaf morphism

$$\begin{aligned} [[r] \mapsto \mathbb{Z}\text{F}(U, X \times (\mathbb{G}_m^{\wedge n})_r)] &\rightarrow [[r] \mapsto \text{Hom}_{sPre_{\bullet}^{fr}(k)}((U \boxtimes \mathbf{G}) / (U \boxtimes +), \mathbb{Z}\text{F}(-, (X \times (\mathbb{G}_m^{\wedge n})_r) \times \mathbb{G}_m^{\wedge 1})) \\ &= \underline{\text{Hom}}_{sPre_{\bullet}(k)}(\mathbb{G}, \mathbb{Z}\text{F}(-, (X \times (\mathbb{G}_m^{\wedge n})_r) \times \mathbb{G}_m^{\wedge 1}))(U)] \end{aligned}$$

given by $\gamma \mapsto \overline{\gamma \boxtimes u_{\mathbb{G}}}$.

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CHEBYSHEV LABORATORY, ST. PETERSBURG STATE UNIVERSITY, 14TH LINE, 29B, 199178 ST. PETERSBURG, RUSSIA

E-mail address: alseang@gmail.com

DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, SINGLETON PARK, SWANSEA SA2 8PP, UNITED KINGDOM

E-mail address: g.garkusha@swansea.ac.uk

ST. PETERSBURG BRANCH OF V. A. STEKLOV MATHEMATICAL INSTITUTE, FONTANKA 27, 191023 ST. PETERSBURG, RUSSIA

ST. PETERSBURG STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS AND MECHANICS, UNIVERSITETSKY PROSPEKT, 28, 198504, PETERHOF, ST. PETERSBURG, RUSSIA

E-mail address: paniniv@gmail.com